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XVIII. *On Dynamical Illustrations of Certain Optical Phenomena.* *By* Professor J. D. Everett, *F.R.S.**

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§ 1. In studying the attempts which have been made to illustrate certain optical phenomena by analogies drawn from chains of particles and from mutually influencing pendulums, I have been impressed with the difficulty of obtaining accurate information respecting the dynamical results in question, and the proofs on which they rest. I have taken some pains to find brief and simple methods of establishing the most important properties, and have here arranged both proofs and results in a form convenient for reference.

In discussing the chain of particles, I have not had recourse to theorems in the Calculus of Finite Differences, but have imitated the methods employed in Lord Kelvin's Baltimore Lectures; and in this connexion I have hit upon a curious geometrical and kinematical theorem which I believe is new.

In dealing with mutually influencing pendulums, whether suspended side by side or one from the other, I have simplified the usual investigation, by first assuming the motion to take place in a fundamental mode, and afterwards discussing combinations of such modes. The deduction of the properties of

* Read March 11, 1898.

the double pendulum here given is, I believe, the fullest to be found anywhere.

A "fundamental mode of vibration" may be defined as one in which *all the particles have simple harmonic vibrations with the same period and either identical or opposite phases.* The resolvability of the most general small permanent vibrations of a system into modes of this simple character is a well-known proposition of abstract dynamics. Anyone who objects to assuming it may regard our plan of operations as a tentative method which is justified when it leads to solutions containing the proper number of arbitrary constants.

§ 2. Suppose a series of equal particles attached at equidistant points to a uniform elastic string of negligible mass. For simplicity we shall ignore gravity, and suppose the only forces acting on each particle to be the tensions of the two portions of string between which it lies. Then the series, when in equilibrium, will lie in a straight line, with uniform tension throughout the whole string. Let F denote this uniform tension, and a the common distance between the particles. Then, in the case of small transverse vibrations in one plane, if y_1, y_2, y_3 denote the displacements of three consecutive particles, and the mass of each particle be M, we shall have

$$\ddot{y}_2 = -\frac{\mathrm{F}}{\mathrm{M}}\frac{y_2 - y_1}{a} - \frac{\mathrm{F}}{\mathrm{M}}\frac{y_2 - y_3}{a},$$

or, putting μ for $\mathrm{F}/\mathrm{M}a$,

$$\ddot{y}_2 = -\mu(y_2 - y_1) - \mu(y_2 - y_3) = -\mu(2y_2 - y_1 - y_3). \quad (1)$$

These formulæ are also applicable to longitudinal vibrations, if we make F denote Young's modulus multiplied by the cross-section.

§ 3. When a simple harmonic undulation is running along the chain of particles, we shall have

$$y = \mathrm{A}\sin 2\pi\left(\frac{x}{\lambda} - \frac{t}{\mathrm{T}}\right), \quad (2)$$

y denoting the displacement of the particle whose undisturbed abscissa is x, λ the wave-length, t the time, T the period, and A the amplitude.

Equation (1) applied to the mth particle gives

$$\ddot{y} = -A\mu \left\{ 2 \sin 2\pi\left(\frac{ma}{\lambda} - \frac{t}{T}\right) - \sin 2\pi\left(\frac{m-1}{\lambda}a - \frac{t}{T}\right) - \sin 2\pi\left(\frac{m+1}{\lambda}a - \frac{t}{T}\right)\right\}$$

$$= -2A\mu \left\{ \sin 2\pi\left(\frac{ma}{\lambda} - \frac{t}{T}\right) - \sin 2\pi\left(\frac{ma}{\lambda} - \frac{t}{T}\right) \cos \frac{2\pi a}{\lambda}\right\}$$

$$= -4A\mu \sin 2\pi\left(\frac{ma}{\lambda} - \frac{t}{T}\right) \sin^2 \frac{\pi a}{\lambda} = -4\mu y \sin^2 \frac{\pi a}{\lambda}.$$

But by direct differentiation of (2), $\ddot{y} = -\left(\frac{2\pi}{T}\right)^2 y$.

Hence

$$\left(\frac{2\pi}{T}\right)^2 = 4\mu \sin^2 \frac{\pi a}{\lambda}; \quad \frac{1}{T^2} = \frac{\mu}{\pi^2} \sin^2 \frac{\pi a}{\lambda}. \quad . \quad . \quad (3)$$

Let V denote the velocity of propagation ; then

$$V^2 = \frac{\lambda^2}{T^2} = \frac{\mu\lambda^2}{\pi^2} \sin^2 \frac{\pi a}{\lambda} = \mu a^2 \left(\frac{\sin \pi a/\lambda}{\pi a/\lambda}\right)^2 \quad . \quad . \quad (4)$$

This is equal to μa^2 when $\lambda = \infty$, and diminishes continuously to zero as $\pi a/\lambda$ increases from 0 to π. Hence $a\sqrt{\mu}$ is the velocity of infinitely long waves, zero is the velocity of waves of length a, and there is continuous diminution of velocity between these limits.

The expression $\frac{\mu}{\pi^2} \sin^2 \frac{\pi a}{\lambda}$ obtained in (3) for the square of the frequency shows that the frequency vanishes for $\lambda = a$ and $\lambda = \infty$, and attains its maximum value $\sqrt{\mu}/\pi$ when $\lambda = 2a$.

The maximum value of the acceleration-factor $-\ddot{y}/y$ or $(2\pi/T)^2$ is 4μ, and is also attained when $\lambda = 2a$. Every frequency less than the maximum corresponds to two different values of λ between a and ∞. Calling them λ_1 and λ_2, we have

$$\frac{a}{\lambda_1} + \frac{a}{\lambda_2} = 1. \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

It thus appears at first sight that two different modes of undulation correspond to each frequency. The following investigation shows that they are merely two different specifications of one and the same motion of the particles.

§ 4. The two undulations

$$y_1 = A \sin 2\pi\left(\frac{x}{\lambda_1} - \frac{t}{T}\right), \quad y_2 = -A \sin 2\pi\left(\frac{x}{\lambda_2} + \frac{t}{T}\right), \quad (6)$$

which have equal amplitudes and opposite directions of propagation, will specify the same motion of the particles if $y_1 - y_2$ vanishes for all values of t when x is a multiple of a, say ma. This gives

$$\sin 2\pi\left(\frac{ma}{\lambda_1} - \frac{t}{T}\right) + \sin 2\pi\left(\frac{ma}{\lambda_2} + \frac{t}{T}\right) = 0,$$

$$\sin \pi\left(\frac{ma}{\lambda_1} + \frac{ma}{\lambda_2}\right) = 0,$$

$$\frac{a}{\lambda_1} + \frac{a}{\lambda_2} = n\,; \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

n denoting any integer either positive or negative. This includes equation (5) as a particular case.

In like manner the two undulations

$$y_1 = A \sin 2\pi\left(\frac{x}{\lambda_1} - \frac{t}{T}\right), \quad y_3 = A \sin 2\pi\left(\frac{x}{\lambda_3} - \frac{t}{T}\right),$$

which have equal amplitudes and the same direction of propagation, will be equivalent if

$$\sin 2\pi\left(\frac{ma}{\lambda_1} - \frac{t}{T}\right) - \sin 2\pi\left(\frac{ma}{\lambda_3} - \frac{t}{T}\right) = 0,$$

$$\sin \pi\left(\frac{ma}{\lambda} - \frac{ma}{\lambda_3}\right) = 0,$$

$$\frac{a}{\lambda_1} - \frac{a}{\lambda_3} = n. \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

One and the same undulating motion of the particles can accordingly be represented in a unlimited number of different ways by the uniform motion of a harmonic curve in the direction of $\pm x$. The amplitude, being the amplitude of the motion of a particle, must be the same for all, but the wave-length may have any value consistent either with equation (7) or equation (8).

The simplest specification is obtained by employing the greatest admissible wave-length. This wave-length, which

we will denote by λ_1, will be unique, and will lie between $2a$ and ∞, except when it is equal to $2a$. In general, the order of magnitude will be λ_1, λ_2, λ_3, &c., as defined by

$$\frac{a}{\lambda_2}=1-\frac{a}{\lambda_1},\quad \frac{a}{\lambda_3}=1+\frac{a}{\lambda_1},\quad \frac{a}{\lambda_4}=2-\frac{a}{\lambda_1},\quad \frac{a}{\lambda_5}+2=\frac{a}{\lambda_1},\quad (9)$$

and so on.

λ_1 is the value obtained by regarding $\pi a/\lambda$ as an angle in the first quadrant.

In the exceptional case $\lambda_1=2a=\lambda_2$, successive particles are in opposite phases, the frequency has the maximum value $\sqrt{\mu/\pi}$, and the acceleration-factor has the maximum value 4μ. The waves in this case can be more simply regarded as stationary.

In the exceptional case $\lambda_1=\infty$, we have $\lambda_2=a=\lambda_3$. The particles are in one straight line, there is no acceleration, and the frequency is zero.

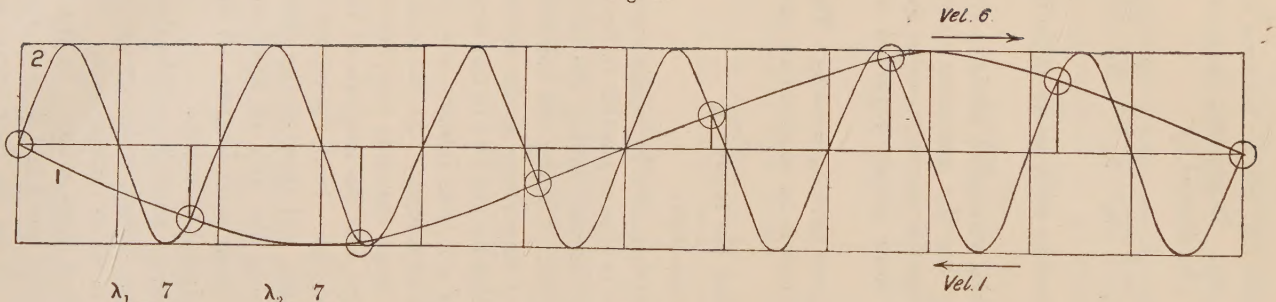
The above investigation establishes the following

Geometrical Proposition.

Through any number of points lying on a harmonic curve and having equidistant ordinates, it is always possible to draw an unlimited number of harmonic curves, having the same amplitude but different wave-lengths. Calling the common distance a, and the wave-length of any curve λ, the curves can be divided into two sets such that for any two of the same set the difference of the values of a/λ is an integer, and for any two of opposite sets the sum is an integer. At the points in question, curves of the same set slope the same way, curves of opposite sets slope opposite ways, and the tangents of the slopes are inversely as the wave-lengths. If all the curves of one set are displaced in one direction, and those of the other set in the opposite direction, along the axis of abscissas, through distances proportional to their wave-lengths, all the curves will still intersect on the original ordinates.

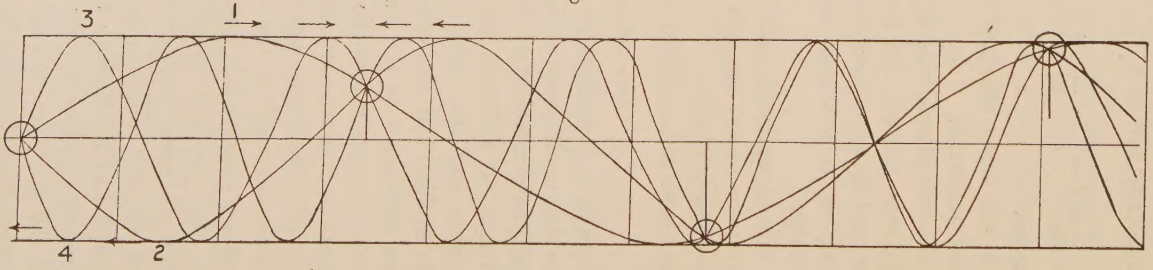
Figs. 1 and 2 illustrate the different possible specifications of the same motion of the particles. Small circles are drawn round the particles to render them more conspicuous. All the curves pass through the centres of these circles. In fig. 1 the values of a/λ for the two curves are $\frac{1}{7}$ and $\frac{6}{7}$, their sum being 1. In fig. 2 the values of a/λ for the four curves are

Fig. 1.



$\frac{\lambda_1}{a} = \frac{7}{1}$. $\frac{\lambda_2}{a} = \frac{7}{6}$.

Fig. 2.



$\frac{a}{\lambda_1} = \frac{2}{5}$. $\frac{a}{\lambda_2} = \frac{3}{5}$. $\frac{a}{\lambda_3} = \frac{7}{5}$. $\frac{a}{\lambda_4} = \frac{8}{5}$.

$\frac{2}{5}$, $\frac{3}{5}$, $\frac{7}{5}$, $\frac{8}{5}$; the difference being 1 for the first and third, and for the second and fourth ; and the sum being 1 for the first and second, 2 for the first and fourth, and 2 for the second and third. In fig. 1 the two curves are supposed to be travelling in opposite directions, the flat curve six times as fast as the steep one. In fig. 2, curves 1 and 3 travel one way, and curves 2 and 4 the opposite way.

§ 5. In our chain of particles, if one particle be constrained to simple harmonic vibration with frequency less than the maximum or *critical* frequency $\sqrt{\mu}/\pi$, the permanent state for the chain will be an undulation whose wave-length is given by

$$\sin \frac{\pi a}{\lambda} = \frac{\pi}{\sqrt{\mu}} \frac{1}{T}, \quad . \quad . \quad . \quad . \quad (3\,A)$$

or a combination of two such undulations travelling in opposite directions ; and this state is one which, if once started, will be kept up by the internal forces of the system.

We shall now investigate the permanent motion of the chain when one of the particles is constrained to simple harmonic vibration of frequency greater than the critical frequency $\sqrt{\mu}/\pi$.

Let y_0 be the displacement at time t of the constrained particle, and $y_1, y_2, y_3 \ldots$ those of the consecutive particles on one side of it. In any "fundamental" or "normal" mode of vibration, the accelerations $\ddot{y}_1$, $\ddot{y}_2$, &c. are proportional to the displacements y_1, y_2, &c. ; so that we may write

$$\ddot{y}_1 = -\omega^2 y_1, \quad \ddot{y}_2 = -\omega^2 y_2, \quad \&c.$$

Hence by equation (1) we have

$$\left.\begin{aligned} \omega^2 y_1 &= \mu(2y_1 - y_0 - y_2), \\ \omega^2 y_2 &= \mu(2y_2 - y_1 - y_3), \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (10)$$

and so on.

Hence

$$\frac{\omega^2}{\mu} - 2 = -\frac{y_0}{y_1} - \frac{y_2}{y_1} = -\frac{y_1}{y_2} - \frac{y_3}{y_2} = \&c. = k \text{ (suppose)}. \quad (11)$$

These equations are satisfied by assuming

$$\left.\begin{aligned} -\frac{y_0}{y_1} = -\frac{y_1}{y_2} = -\frac{y_2}{y_3} = \&c. &= r, \\ k &= r + \frac{1}{r}, \end{aligned}\right\} \quad . \quad . \quad . \quad (12)$$

if k is greater than 2, or if $\omega^2 > 4\mu$, or (since ω is $2\pi/T$) if $1/T > \sqrt{\mu}/\pi$.

The same result can be deduced by writing equations (11) in the form

$$-\frac{y_0}{y_1} = k + \frac{y_2}{y_1}, \quad -\frac{y_1}{y_2} = k + \frac{y_3}{y_2}, \quad -\frac{y_2}{y_3} = k + \frac{y_4}{y_3}, \quad \&c.; \qquad (13)$$

whence we obtain for any one of the ratios $-\frac{y_0}{y_1}$, $-\frac{y_1}{y_2}$, &c. the value

$$k - \cfrac{1}{k - \cfrac{1}{k - \&c.}} = r \text{ (suppose)}. \quad . \quad . \quad . \quad . \quad (14)$$

That is,

$$r = k - \frac{1}{r},$$

which is the assumption employed above. The assumption gives a choice between two values of r, one being the reciprocal of the other. We must clearly choose the value which gives a decreasing not an increasing geometrical progression. r and its reciprocal are the roots of the equation

$$r^2 - kr + 1 = 0.$$

Hence we have

$$2r = k + \sqrt{(k^2-4)}, \quad \frac{2}{r} = k - \sqrt{(k^2-4)}, \quad . \quad . \quad (15)$$

where

$$k = \frac{\omega^2}{\mu} - 2, \quad \sqrt{(k^2-4)} = \frac{\omega}{\mu}\sqrt{(\omega^2 - 4\mu)}.$$

Thus the resulting motion is completely determined. Successive particles will be opposite in phase, and their amplitudes will diminish in geometrical progression as we move away from the particle which is subjected to external constraint, the diminution being the more rapid as the frequency is greater.

§ 6. The constraining force (taking the mass of the particle as unity) is the excess of the actual acceleration $-\omega^2 y_0$ above the acceleration $-2\mu(y_0 - y_1)$ or $-2\mu y_0\left(1 + \frac{1}{r}\right)$ due to the tensions. It is therefore $y_0\left(-\omega^2 + 2\mu + \frac{2\mu}{r}\right)$.

But, from above,

$$\frac{2\mu}{r} = \omega^2 - 2\mu - \omega\sqrt{(\omega^2 - 4\mu)}.$$

Hence the constraining force is

$$-y_0\omega\,(\omega^2-4\mu). \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

It is proportional and opposite to the displacement y_0, and therefore does equal amounts of positive and negative work. These remarks apply to the permanent regime only.

The critical value of ω which separates free from forced vibrations is $2\sqrt{\mu}$, and this value makes the constraining force zero. As ω increases from this minimum to infinity, the constraining force increases from zero to infinity, and the ratio r of successive amplitudes increases from unity to infinity.

The work done in the initial stage is represented by the energy of the chain in the permanent state. The ratio of the energy of the whole chain (extending to infinity on both sides) to the energy of the particle to which the constraint is applied is

$$1+2\left(\frac{1}{r^2}+\frac{1}{r^4}+\frac{1}{r^6}+\&c.\right)=\frac{r^2+1}{r^2-1},$$

which by (15) reduces to

$$\frac{k\{k+\sqrt{(k^2-4)}\}}{\sqrt{(k^2-4)}\,\{k+\sqrt{(k^2-4)}\}}=\frac{k}{\sqrt{(k^2-4)}}=\frac{\omega^2-2\mu}{\sqrt{\{\omega^2(\omega^2-4\mu)\}}}.$$

When ω has its least value $2\sqrt{\mu}$ this is infinite, as it ought to be, since we have in this case an infinite trail of equal oscillations. As ω increases, the value steadily diminishes to the limit unity, showing that the frequency may be so great as practically to confine the energy to the first particle.

§ 7. This discussion of the simple harmonic vibrations of a chain of particles serves to explain Sir George Stokes's illustration of fluorescence, as quoted in Tait's ' Light,' pp. 161–163.

When the frequency of the ætherial vibrations is below the critical value, any nascent disturbance is carried off to a distance by undulations ; but when it is above the critical value the effects accumulate at the origin of the disturbance. In the latter case, when the applied force ceases to act, the subsequent motion is compounded of *free* simple harmonic vibrations ; and for none of these does the frequency exceed

the critical value. Hence there is a change from higher to lower frequency, and therefore a lessening of refrangibility.

In support of the view that there is a critical frequency for a fluorescent substance, Sir G. Stokes says :—

"In dealing with a single fluorescent substance—not a mixture of two or more—I have generally found that the following feature is (very approximately, at any rate) observed :—As we take incident light of increasing refrangibility, it is at first inactive ; then, on reaching a certain point P of the spectrum, it begins to produce fluorescence, and the heterogeneous fluorescent light contains refrangibilities not extending beyond P. As we continue to progress in the incident spectrum, the highest refrangibility of the fluorescent light does not follow the refrangibility of the incident light, but remains about P."

Professor Preston maintains that there is no difference in kind between fluorescence and the process by which lamp-black transforms luminous into non-luminous radiation. In the application of our analogy to lamp-black the critical frequency will be below the range of visibility.

§ 8. The fundamental modes of *stationary* vibration for the chain are most easily deduced from the consideration of two travelling undulations. If the fixed ends coincide with two of the particles, and the intervening length is ma, the equation for a fundamental mode is

$$y = 2\mathrm{A} \sin 2\pi x/\lambda \cos 2\pi t/\mathrm{T},$$

λ being either $2ma$ or any submultiple of it exceeding $2a$. This gives $m-1$ modes.

As an example, if the distance between the fixed ends is $4a$, the values of λ are

$$8a, \qquad 4a, \qquad 8a/3.$$

In the first mode the amplitudes of the three free particles are as the sines of

$$\pi/4, \qquad \pi/2, \qquad 3\pi/4.$$

In the second mode, as the sines of

$$\pi/2, \qquad \pi, \qquad 3\pi/2.$$

In the third mode, as the sines of

$$3\pi/4, \qquad 3\pi/2, \qquad 9\pi/4.$$

§ 9. Another mechanical illustration that has been often mentioned as analogous to certain optical phenomena is the mutual influence of pendulums.

First, suppose two simple pendulums of masses 1 and s, with natural frequencies $\omega_1/2\pi$ and $\omega_2/2\pi$, their bobs being at the same level, and elastically connected so that there is a mutual push or pull according as their distance is less or greater than when both pendulums are vertical. The connexions are supposed to be of negligible mass, and the vibrations to be so small that the vertical component of the push or pull is negligible. The movements are supposed to be confined to one vertical plane.

Let x denote displacement of the mass 1 from the vertical through its point of support, reckoned positive when towards s, and ξ the displacement of the mass s, reckoned positive when away from the mass 1. Then $x-\xi$ is their approach, and $\xi-x$ their recess. Let $\mu(x-\xi)$ be the push, and $\mu(\xi-x)$ the pull.

If the masses were unconnected, we should have

$$\ddot{x} = -\omega_1^2 x, \qquad \ddot{\xi} = -\omega_2^2 \xi. \quad . \quad . \quad . \quad . \quad (17)$$

When they are connected, we have

$$\left.\begin{aligned} \ddot{x} &= -\omega_1^2 x + \mu(\xi - x), \\ \ddot{\xi} &= -\omega_2^2 \xi + \frac{\mu}{s}(x - \xi), \end{aligned}\right\} . \quad . \quad . \quad . \quad . \quad (18)$$

When the system is vibrating in a fundamental mode, x and ξ are in a constant ratio. Assume $\xi = kx$; then equations (18) become

$$\left.\begin{aligned} \ddot{x} &= -\{\omega_1^2 + \mu(1-k)\}x, \\ k\ddot{x} &= -\left\{\omega_2^2 + \frac{\mu}{s}\left(1 - \frac{1}{k}\right)\right\} kx, \end{aligned}\right\} . \quad . \quad . \quad (19)$$

By division and reduction, we have the quadratic in k,

$$\mu k^2 + k\left(\omega_2^2 - \omega_1^2 + \frac{\mu}{s} - \mu\right) - \frac{\mu}{s} = 0, \quad . \quad . \quad . \quad (20)$$

which has a positive root k_1 and a negative root $-k_2$.

Let the corresponding values of $2\pi/T$ for the fundamental modes be called Ω_1 and Ω_2. For both of them we have,

by equations (19),

$$\Omega^2 = \omega_1^2 + \mu(1-k) = \omega_2^2 + \frac{\mu}{s}\left(1-\frac{1}{k}\right). \quad . \quad . \quad . \quad (21)$$

Putting $-k_2$ for k, we have

$$\Omega_2^2 = \omega_1^2 + \mu(1+k_2) = \omega_2^2 + \frac{\mu}{s}\left(1+\frac{1}{k_2}\right), \quad . \quad . \quad (22)$$

showing that Ω_2 is greater than either ω_1 or ω_2; in other words, that the fundamental mode in which the displacements are opposite has a higher frequency than the vibration of either pendulum alone.

Putting k_1 for k, we have

$$\Omega_1^2 = \omega_1^2 + \mu(1-k_1) = \omega_2^2 + \frac{\mu}{s}\left(1-\frac{1}{k_1}\right). \quad . \quad . \quad . \quad (23)$$

When k_1 is different from unity, one of the two quantities $1-k_1$ and $1-1/k_1$ is positive and the other negative; hence Ω_1 is intermediate between ω_1 and ω_2. Equation (20) shows that, if k_1 is unity, ω_1 is equal to ω_2; that is, the pendulums are of equal length. Whenever they are unequal, the fundamental mode in which their displacements are similar is intermediate in frequency between the vibrations of the two pendulums singly.

The actual vibration of the system will be either that corresponding to k_1 and Ω_1, or that corresponding to $-k_2$ and Ω_2, or a combination of the two in an arbitrary ratio; according to the initial circumstances.

If the pendulums are nearly equal, both in length and mass, the coefficient $\omega_2^2 - \omega_1^2 + \mu/s - \mu$ of k in the quadratic is small and the positive and negative roots are nearly equal, also their product $-1/s$ is -1 nearly; hence the roots are approximately ± 1. The values of Ω^2 from (21) are therefore approximately

$$\Omega_1^2 = \omega_1^2, \quad \Omega_2^2 = \omega_1^2 + 2\mu,$$

the latter being always the greater, as proved above.

In practice μ is usually exceedingly small; hence Ω_1 and Ω_2 are nearly equal. The general equations

$$\left.\begin{aligned} x &= \mathrm{A}\cos(\Omega_1 t - \alpha_1) + \mathrm{B}\cos(\Omega_2 t - \alpha_2), \\ \xi &= k_1\mathrm{A}\cos(\Omega_1 t - \alpha_1) - k_2\mathrm{B}\cos(\Omega_2 t - \alpha_2), \end{aligned}\right\} \quad . \quad (24)$$

will accordingly represent what are called in acoustics "curves of beats," the beats being strongest when A and B (and therefore also k_1A and k_2B) are nearly equal.

If we start from a condition in which the pendulum x is at rest in the zero position, the equations will be

$$\left.\begin{aligned} x &= \mathrm{A}(\cos\Omega_1 t - \cos\Omega_2 t), \\ \xi &= \mathrm{A}(k_1\cos\Omega_1 t + k_2\cos\Omega_2 t) \\ &= \mathrm{A}(\cos\Omega_1 t + \cos\Omega_2 t), \text{ nearly,} \end{aligned}\right\} \quad . \quad . \quad (25)$$

When t is such that $\cos\Omega_1 t$ and $\cos\Omega_2 t$ are each sensibly unity, the first pendulum will have sensibly zero excursions, and the second pendulum will have maximum excursions. When one of the cosines is sensibly 1, and the other sensibly -1, these conditions will be reversed. In fact, we have

$$\left.\begin{aligned} x &= 2\mathrm{A}\sin\frac{\Omega_2-\Omega_1}{2}t\,.\,\sin\frac{\Omega_2+\Omega_1}{2}t, \text{ rigorously,} \\ \xi &= 2\mathrm{A}\cos\frac{\Omega_2-\Omega_1}{2}t\,.\,\cos\frac{\Omega_2+\Omega_1}{2}t, \text{ nearly,} \end{aligned}\right\} \quad . \quad (26)$$

showing that the excursions are

$$2\mathrm{A}\sin\tfrac{1}{2}(\Omega_2-\Omega_1)t \quad \text{and} \quad 2\mathrm{A}\cos\tfrac{1}{2}(\Omega_2-\Omega_1)t,$$

each of which in its turn vanishes when the other is 2A. This can be illustrated by hanging two equal pendulums from the same stand. As regards phases, the comparison of the two factors

$$\sin\tfrac{1}{2}(\Omega_2+\Omega_1)t \quad \text{and} \quad \cos\tfrac{1}{2}(\Omega_2+\Omega_1)t$$

shows that, at first, ξ is earlier than x by a quarter period.

§ 10. Next take the case of one pendulum suspended from another, each consisting of a heavy particle at the end of a weightless thread.

Upper mass	1,	Lower mass	s ;
Length of upper pendulum	a,	of lower	b ;
Displacement of upper mass	x,	of lower	ξ ;

the displacements being measured horizontally from a vertical through the fixed point of support, and being so small that vertical accelerations may be neglected. Then, since the

tensions are sg and $(1+s)g$, we have

$$\left.\begin{aligned} s\ddot{\xi} &= -sg\frac{\xi-x}{b}, \qquad \ddot{\xi} = -\frac{g}{b}(\xi-x), \\ \ddot{x} &= sg\frac{\xi-x}{b} - (1+s)g\frac{x}{a} = \frac{sg}{b}\xi - g\left(\frac{s}{b}+\frac{1+s}{a}\right)x. \end{aligned}\right\} \quad . \quad (27)$$

Assume $\xi = kx$ for a fundamental mode. Then the equations become

$$\left.\begin{aligned} k\ddot{x} &= -\frac{g}{b}(k-1)x \\ \ddot{x} &= -\frac{g}{b}\left\{s+\frac{b}{a}(1+s)-ks\right\}x \end{aligned}\right\} \quad . \quad . \quad (28)$$

By division and reduction, we obtain the quadratic for k

$$ask^2 - k\{b-a+(b+a)s\} - a = 0\,; \quad . \quad . \quad . \quad (29)$$

which, as in the previous example, has a positive root k_1 and a negative root $-k_2$

As before, denoting the acceleration-factor $-\ddot{x}/x$ or $(2\pi/\mathrm{T})^2$ by Ω^2, the two equations give

$$\Omega^2 = \frac{g}{b}\left(1-\frac{1}{k}\right) = \frac{g}{a}\left\{1+s+\frac{as}{b}(1-k)\right\}. \quad . \quad . \quad (30)$$

Putting $-k_2$ for k all the terms are positive ; hence the first expression for Ω_2^2 is greater than g/b, and the second is greater than g/a ; that is to say :—*The mode in which the displacements are opposite is quicker than either pendulum alone.*

For the other mode, we have

$$\Omega_1^2 = \frac{g}{b}\left(1-\frac{1}{k_1}\right) = \frac{g}{a}\left\{1+s+\frac{as}{b}(1-k_1)\right\}. \quad . \quad . \quad (31)$$

The first value shows that Ω_1^2 is less than g/b. The second value shows that it will be less than g/a if $s+(1-k_1)\,as/b$ is negative, that is if k_1 is greater than $1+b/a$; and this condition is always fulfilled, for the substitution of this value of k in the quadratic gives an opposite sign to the substitution of a very large positive quantity. Hence *the mode in which the displacements are similar is slower than either pendulum alone.*

The physical meaning of the result $k_1 > 1+b/a$ is that, in

the mode in which the displacements are similar, the lower string is more inclined to the vertical than the upper.

When a is infinite, $-k_2$ is $-1/s$, and Ω_2^2 is $(1+s)g/b$.

When b is infinite, k is zero, and Ω^2 is $(1+s)g/a$.

In the former case, the lower string rotates about the common centre of gravity as a fixed point, so that the virtual length of the pendulum is $b/(1+s)$. In the latter, the lower string simply alters the downward force on the upper mass from g to $(1+s)g$.

§ 11. When the ratio s of the lower to the upper mass is very small, and a and b are not approximately equal, the quadratic (29) may be written

$$k^2+k\frac{a-b}{as}-\frac{1}{s}=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (32)$$

Neglecting the first term (which is small compared with the others), we have

$$k=\frac{a}{a-b},$$

as the approximate value of one root. It is the value obtained by putting $s=0$ in the general quadratic, and is identical with the value of k obtained by supposing the point of support constrained to vibrate like a pendulum of length a. The other approximate root is

$$k=\frac{b-a}{as};$$

for the product of the two roots must be $-1/s$.

It will be noted that the first root makes the excursions of the two pendulums comparable with one another; whereas the second root makes the excursions of the upper very small compared with the lower.

The closest approximations to the values of Ω^2 are got by employing in connexion with the first root the formula

$$\Omega^2=\frac{g}{a}\left\{1+s+\frac{as}{b}(1-k)\right\},$$

and in connexion with the second root the formula

$$\Omega^2=\frac{g}{b}1-\frac{1}{k}.$$

The specifications of the two modes will accordingly be

First Mode.	Second Mode.
$k=\frac{a}{a-b}$,	$k=\frac{b-a}{sa}$,
$\Omega^2=\frac{g}{a}\left(1+\frac{sb}{b-a}\right)$.	$\Omega^2=\frac{g}{b}\left(1+\frac{sa}{a-b}\right)$.

. . . (33)

The first mode nearly agrees in period with a pendulum of length a, and the second with a pendulum of length b.

The two values of k are obviously opposite in sign. The positive k always makes Ω^2 less than the lesser of g/a and g/b, and the negative k makes it greater than the greater, in accordance with the general rule. When a is infinite one value of Ω^2 is $(1+s)g/b$, and when b is infinite one value is $(1+s)g/a$, as previously found for the general case.

§ 12. The results obtained in the preceding section agree with the statements contained in Lord Kelvin's paper* "On the Rate of a Clock or Chronometer as influenced by the Mode of Suspension," some of which are quoted in recent editions of Tait and Steele's 'Dynamics.'

Lord Kelvin selects the second mode as the practical mode for a pendulum supported on a yielding stand, presumably because the amplitude of the pendulum in the first mode is comparable with that of the point of support and therefore inappreciable. He also selects it as the practical mode for an ordinary spring clock suspended like a compound pendulum from a fixed axis; presumably because the escapement would not work with such an enormous departure from correct time as the first mode would involve.

On the other hand, Lord Rayleigh, in a paragraph which is often quoted in discussions on anomalous dispersion, ignores the second mode and adopts the first. The justification seems to be that his argument relates to the behaviour of the upper pendulum, and that in the second mode the excursions of the upper pendulum are infinitesimal.

In the first mode, the ratio of the amplitude of the upper to that of the lower pendulum is $1/k=(a-b)/a$, which may be small, but not negligible, and the ratio of the changed

* Glasgow, Trans. Inst. Engin. x. 1867, pp. 139–150.

to the natural (frequency)^2 for the upper is $1+sb/(b-a)$. The upper pendulum is therefore quickened if $b-a$ is positive, that is, if the lower pendulum is naturally the slower, and is retarded if the lower pendulum is naturally the quicker. Lord Rayleigh suggested many years ago that the paradoxical appearances presented in "anomalous dispersion" are due to an action analogous to this. In anomalous dispersion there is always excessively strong selective absorption, evidenced by a black band in the spectrum produced by a prism of the anomalous substance; and the colours which are not thus blotted out are displaced from their usual order. A general statement of the facts, which has been deduced from the comparison of a number of anomalous spectra, is that, in passing through the colours of the ordinary spectrum from red to violet, the refractive index is increased where the absorption increases rapidly, and diminished where absorption diminishes rapidly. In the case of iodine vapour, red is thus rendered more refrangible than blue and violet.

According to Lord Rayleigh, we are to regard the vibrating æther as analogous to the heavy upper pendulum, and the vibrating molecules of the substance to the light lower pendulum. The colour which is absorbed is the colour which has the same frequency as the molecules of the substance. Colours which, in the ordinary spectrum, are on the red side of the absorbed colour have lower frequency than the molecular vibrations. This is the case of "upper pendulum naturally the slower," and the resultant vibration will be slower still. Lord Rayleigh calls this effect an increase in the "virtual inertia" of the upper pendulum. I subjoin Lord Rayleigh's own words (Phil. Mag. xliii. p. 322), which are quoted in full in Lord Kelvin's 'Baltimore Lectures,' and are paraphrased in Preston's 'Light.' Professor Preston identifies the "pendulum" in the quotation with the lower of the two, and the "point subject to horizontal vibration" with the bob of the upper; and this is also my own understanding of the passage.

"The effect of a pendulum, suspended from a point subject to horizontal vibration, is to increase or diminish the virtual inertia of the mass, according as the natural period of the

pendulum is shorter or longer than that of its point of suspension. This may be expressed by saying that, if the point of support tends to vibrate more rapidly than the pendulum, it is made to go faster still, and *vice versâ*.

"Below the absorption band, the material vibration is naturally higher, and hence the effect of the associated matter is to increase (abnormally) the virtual inertia of the æther, and therefore the refrangibility. On the other side the effect is the reverse."

The latter part of the passage is to me somewhat obscure. The analogy seems to point to a change of frequency, but instead of this we have a change in velocity of propagation.

§ 13. When $a-b$ approaches zero, the foregoing approximation is insufficient, and the following investigation is preferable.

If the coefficient of k in the quadratic (29) vanishes, we have

$$\left.\begin{aligned} s&=\frac{a-b}{a+b}, & k&=\pm\frac{1}{\sqrt{s}},\\ \Omega_1^2&=\frac{g}{b}(1-\sqrt{s}), & \Omega_2^2&=\frac{g}{b}(1+\sqrt{s}). \end{aligned}\right\} \quad . \quad . \quad . \quad (34)$$

As s, being the ratio of the masses, cannot be negative, these conditions require $a-b$ to be positive, and s will lie between the limits 0 and 1. We shall have

$$\left.\begin{aligned} x&=\mathrm{A}\cos\Omega_1 t+\mathrm{B}\cos(\Omega_2 t-\alpha),\\ \xi&=\frac{1}{\sqrt{s}}\{\mathrm{A}\cos\Omega_1 t-\mathrm{B}\cos(\Omega_2 t-\alpha)\}\ ; \end{aligned}\right\} \quad . \quad . \quad . \quad (35)$$

and if the initial values of x, ξ, and $\dot{x}$ are zero, we have $\alpha=0$, $\mathrm{A}=-\mathrm{B}$; hence

$$\left.\begin{aligned} x&=\mathrm{A}(\cos\Omega_1 t-\cos\Omega_2 t),\\ \xi&=\frac{\mathrm{A}}{\sqrt{s}}(\cos\Omega_1 t+\cos\Omega_2 t). \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (36)$$

If s is small, Ω_1 and Ω_2 are nearly equal, and each pendulum oscillates with amplitude varying between zero and a maximum, the maximum of one coinciding with the zero of the other. The maximum amplitude is 2A for x, and $2\mathrm{A}/\sqrt{s}$

for ξ. The phases are, at first, a quarter-period earlier for ξ than for x. (See equations (26).)

If the initial conditions are altered by making ξ vanish instead of x, the plus and minus in the expressions for x and ξ will be interchanged, and the phases will, at first, be a quarter-period earlier for x than for ξ, the equations being now reducible to the forms

$$\left.\begin{aligned} x &= 2\mathrm{A}\cos\tfrac{1}{2}(\Omega_2-\Omega_1)t\,.\,\cos\tfrac{1}{2}(\Omega_2+\Omega_1)t, \\ \xi &= \frac{2\mathrm{A}}{\sqrt{s}}\sin\tfrac{1}{2}(\Omega_2-\Omega_1)t\,.\,\sin\tfrac{1}{2}(\Omega_2+\Omega_1)t. \end{aligned}\right\} \quad . \quad . \quad (37)$$

This investigation applies to the experiment described in Rayleigh on Sound, 2nd edition § 62 ; but the experiment appears to have been stopped as soon as the lower pendulum attained its first maximum.

§ 14. Sellmeier (Pogg. *Ann.* vol. cxlv. p. 534) refers to the transference of energy which goes on from one pendulum to the other when the amplitudes vanish alternately, and maintains that a similar transference of energy between particles embedded in the æther and particles of a fluorescent body is the cause of fluorescence. He works out the case in which s is small, $a=b$, and one of the pendulums is initially at rest in the zero position.

XIX. *On the Circulation of the Residual Gaseous Matter in a Crookes Tube.* *By* Alan A. Campbell Swinton*.

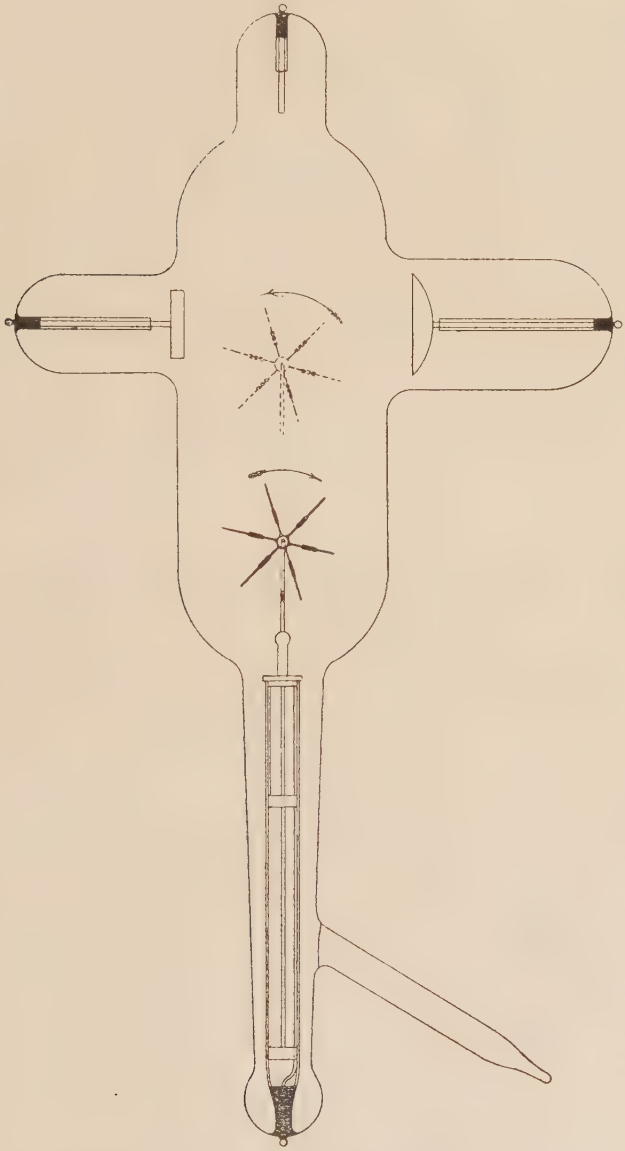
There appears to be generally some doubt as to the way in which the particles or atoms that form the cathode-rays in a Crookes tube return again to the cathode. It is obvious that they must return, as otherwise they would all become collected at one end of the tube, and the cathode would soon be surrounded by an absolute vacuum. By some it has been supposed that the atoms return by ordinary diffusion in the intervals between the succeeding electrical discharges; by some that they creep back along the inner surface of the glass walls of the tube; by others that they return during a discharge through the space between the cathode-rays and the glass. Further, there is the question whether the returning atoms carry any portion of the positive electricity from the anode to the cathode, similarly as part at any rate of the negative electricity is believed to be carried from the cathode to the anodic portion of the tube by the cathode-ray atoms.

In his 1891 Presidential Address to the Institution of Electrical Engineers, Sir William Crookes described a tube which was divided into two halves by a diaphragm pierced with two small apertures. Near each aperture was mounted a small wheel with vanes to detect and show the direction of any stream of atoms that might pass through. It was found that when the cathode was caused to project rays through one aperture, the rotation of the wheel at the other aperture showed the atoms in the act of returning.

In this tube, however, both anode and cathode were on the same side of the diaphragm, the anode being behind the cathode. So the experiment cannot be said to decide the existence of a true anode-stream, but merely to demonstrate that the action of the cathode-stream was to create a difference of gaseous pressure in the two halves of the tube which relieved itself by a current of atoms through the spare aperture in the diaphragm.

* Read March 25, 1898.

Fig. 1.



The writer, with the assistance of Mr. J. C. M. Stanton and Mr. H. Tyson Wolff, has investigated the matter further by means of a series of tubes, one of which is illustrated in

fig. 1. In this tube, which is very highly exhausted, we have as two electrodes a concave aluminium cup and an aluminium plate placed opposite to one another, as in an ordinary focus-tube. As will be seen, there is also a supplemental wire electrode at one side. Inside the tube is a very delicately pivoted radiometer-wheel with mica vanes, which is so mounted on a sliding-rod that by simply shaking the tube the wheel can be moved out into the centre, as indicated by the dotted lines, so that the cathode-stream impinges directly upon the vanes, or can be moved back to the position shown in full lines in the illustration, when the vanes are quite out of the cathode-stream. When the wheel is in the former position the tube acts exactly as an ordinary Crookes electric radiometer-tube, the wheel rotating one way or another in the direction of the cathode discharge as the concave cup or the flat-plate electrode is made cathode, the rotation being much more rapid in the latter than in the former case.

On the other hand, if the wheel is moved well out of the cathode-stream, provided the exhaustion is high enough, it is found to rotate in the opposite direction to the cathode-rays, that is to say, in the direction that indicates an atomic stream from anode to cathode round the outside of the cathode-stream. The rotation will only take place with high exhaustion, and is never so rapid as in the previous case when the vanes were in the cathode-stream; but it is faster and faster the higher the exhaustion, and at very high vacua the speed is very considerable. The wheel rotates whether the concave or the flat plate is made cathode, but in either case in a direction opposite to what it did when in the cathode-stream.

Very little electric power is necessary to show this effect. With a sufficiently high vacuum a small Wimshurst machine, passing so little current through the tube that scarcely any fluorescence is visible, will cause the wheel to rotate at a speed of many turns per second; and rotation can even be produced by the mere approach of two oppositely charged leyden-jars to the terminals of the tube.

Further experiment appears to show that the rotation is really due to a stream returning round about the cathode-stream from anode to cathode, and that the atoms or particles

of which this stream consists are positively charged. In the experiments so far described the anode was also the anti-cathode; and it is obvious that the stream might not be truly an anode-stream, but merely a reflected or splashed cathode-stream. It is, however, found that if the flat plate used as anode be earthed, or if the plate be disconnected from the electrical source of power, and used merely as an anti-cathode, the spare electrode already alluded to opposite to the wheel being employed as anode, the wheel refuses to rotate, though rotation recommences immediately in the first case when the earth-connexion is removed, and in the latter case when the anode and anti-cathode are connected together. This appears to be fairly conclusive evidence that it is a stream projected from the anode that causes the rotation of the wheel. It might, however, be argued that this stream, though coming from the anode, is in reality a cathode-stream due to oscillations in the electric discharge causing the anode to be at times negatively charged. Against this view it should be stated that the effects are produced quite as well by the silent discharge from a small Wimshurst machine as with Ruhmkorff-coil discharges. It has further been noted that the wheel rotates in the proper direction with either the flat plate or the concave cup employed as cathode, the other being used as anode; and any cathode-rays given off by the concave cup would be concentrated, and could not therefore reach the wheel in the position in which it is used to show the stream from the anode. Again, it might be supposed that the rotation is due to heat-radiation from the anti-cathode. In this case, however, the rotation should occur when the side anode is employed, which, as mentioned, is not the case. Further, even when such powerful discharges are used as to make the anti-cathode visibly red hot, the rotation of the wheel ceases almost immediately the current is interrupted, the movement that continues being obviously due to the momentum of the wheel.

The experiments therefore appear to establish the existence at high exhaustions of a true anode-stream which travels from the anode to the cathode, just as does the cathode-stream from the cathode to the anti-cathode, the anode-stream passing round the exterior of the cathode-stream at a considerably

lower velocity than the latter, but at a greater and greater velocity the higher the exhaustion.

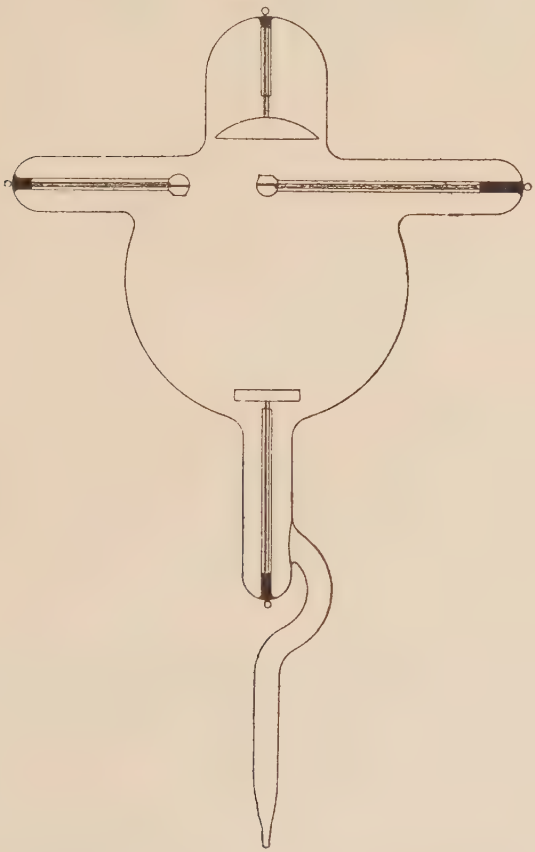
It also appears that while, as is well known, the cathode-stream is negatively charged, the anode-stream is charged positively.

For the purpose of ascertaining this a tube fitted with exploring-poles, as used by Sir William Crookes, was employed. These poles were, however, somewhat differently arranged to any described in Sir W. Crookes's papers, while the exhaustion, which was taken to the degree required for Röntgen-ray work in which the negative dark space appears to fill the whole tube, was probably much higher than that Sir W. Crookes employed. Such a tube is illustrated in fig. 2, and contains the usual aluminium cathode-cup and anode-plate. The exploring-poles consist of aluminium wires tipped with platinum plates enclosed in glass tubes, which are blown out into small bulbs at the free extremities so as to contain and shield the platinum plates, the bulbs having, however, each an aperture exposing the platinum on one side. One platinum plate is arranged just opposite to the centre of the cathode-cup, and the aperture in its containing glass bulb faces the cathode so that the cathode-rays can impinge upon the platinum. The other and shorter pole has its platinum plate well to one side of the cathode, and has the aperture in its glass cup turned away from the cathode towards the anode. Experiments were conducted with this tube highly exhausted and excited by means of an induction-coil, the polarity of the two exploring-poles being ascertained by means of a quadrant-electrometer. With the aluminium cup as cathode and the plate as anode, the longer exploring-pole, which has its bare extremity facing the cathode-stream, was found invariably to be charged negatively, while the other and shorter exploring-pole was found always to be charged positively. This was found to be more and more distinctly the case the higher the exhaustion.

It seems therefore that at high vacua at any rate some portion of the positive electricity passing through the tube is carried by the positively charged atoms or particles that form the anode-stream. Very probably at lower exhaustions the electric discharge passes through the tube chiefly by an inter-

change of electrical charges from molecule to molecule on the Grothüss chain principle. At very high exhaustions, however, when the mean free path becomes considerable, this may cease to be the case, at any rate to a large degree; and there may

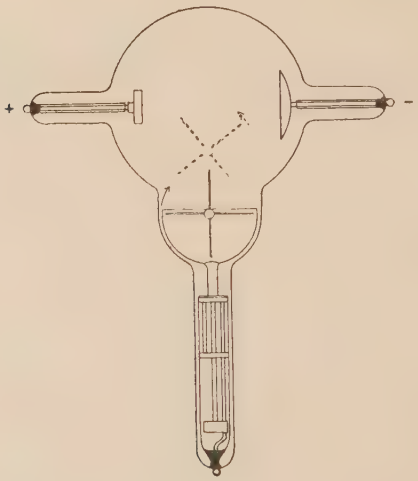
Fig. 2.



be to some extent a regular and complete circulation of the positively and negatively charged atoms, some of which may make the entire journey from anode to cathode, or *vice versâ*, and deliver up their charges not by interchange with other gaseous atoms, but by direct convection to the electrodes of opposite sign.

It may be mentioned that for showing the movements of the streams a tube of the form illustrated in fig. 1 is not essential, the anode-stream being equally well marked in a tube of the ordinary globular form, provided the wheel is mounted so as to be half contained within a glass cup arranged so as to prevent the stream acting equally and oppositely upon the vanes upon diametrically opposite sides of the wheel, as shown in fig. 3. Further, it is not necessary to employ the

Fig. 3.



sliding adjustment for the wheel, as the effects can be shown equally well by means of two separate tubes, one with its wheel in the forward position and the other with its wheel in the back position. The two tubes can be operated simultaneously by connecting them in series.

It should also be stated that for these experiments extremely high vacua are requisite, and that with a Ruhmkorff coil as the source of electrical power the effects can only be shown satisfactorily with the tube connected to the mercury-pump, for the reason that the discharges from the coil inevitably bring down the vacuum very quickly, apparently by their action upon the mica vanes, which are visibly affected when the cathode-stream from the concave cup is allowed to fall upon them. Using a small Wimshurst machine, however,

the effects can be shown after the tube has been sealed off, though even then with use the vacuum appears to deteriorate in a short time.

Discussion.

Prof. Boys said he did not feel altogether convinced by the experiments that the rotation of the mill was due to simple mechanical motion of the particles of matter between the electrodes. The weight of air left in the tube at such high degrees of exhaustion was extremely small; it was difficult to realize that its impact could produce the sudden mechanical effect observed at the moment of the reversal of the rotation of the mill.

Mr. Wimshurst thought it important to keep in mind the existence of mercury vapour in the tube. He also referred to some experiments in which a bar of metal was used to explore a focus tube, by observations of the changes of luminosity produced in different positions.

Dr. Chree said that if the rotations of the mill could be shown to indicate a velocity of the particles, of the same order as that observed in Crookes's experiments, it was safe to assume the existence of a similar cause. This might be important in deciding as to the general truth of the bombardment theory of Crookes. He asked whether the rotation had been investigated within the dark space around the cathode.

Mr. Appleyard suggested that in tracing the cause of the rotation, it would lead to simpler results if the vanes of the mill were made of some light conducting substance. Mica introduced difficulties, owing to its retention of the charges.

Prof. Boys pointed out that this could be done by gilding the mica.

Mr. Campbell Swinton, in reply, said that the objection raised by Prof. Boys to the mechanical theory of the rotation would apply equally to the whole theory of electro-radiometry, including the case of the mill used originally by Crookes in the direct path of the cathode stream. But it must be remembered that, although the mass of matter present within the tube was very small, its velocity was

proportionally great; it was of the order of 9000 kilometres per second. Hence, the contained matter might be conceived as capable of producing the observed acceleration, and Crookes's bombardment theory might safely be adopted as a good working hypothesis. In the tubes used for these experiments, the exhaustion was carried so high that the negative dark space appeared to fill the whole tube. He had, so far, only tried mica for the vanes, but he thought it would be important to observe the results with a substance that did not retain the charges.

XX. *Some Further Experiments on the Circulation of the Residual Gaseous Matter in Crookes Tubes.* *By* Alan A. Campbell Swinton*.

In the discussion which followed my former paper on this subject some objection was taken to the use of a non-conducting substance, *i. e.* mica, for the vanes of the mill which was used to detect the circulation of the ultra-gaseous matter, it being suggested by Professor Boys that the rotation produced might be the result of electrification of the vanes. It was further suggested by Mr. Appleyard that gilding the vanes, so as to make them conductive, might modify the effect.

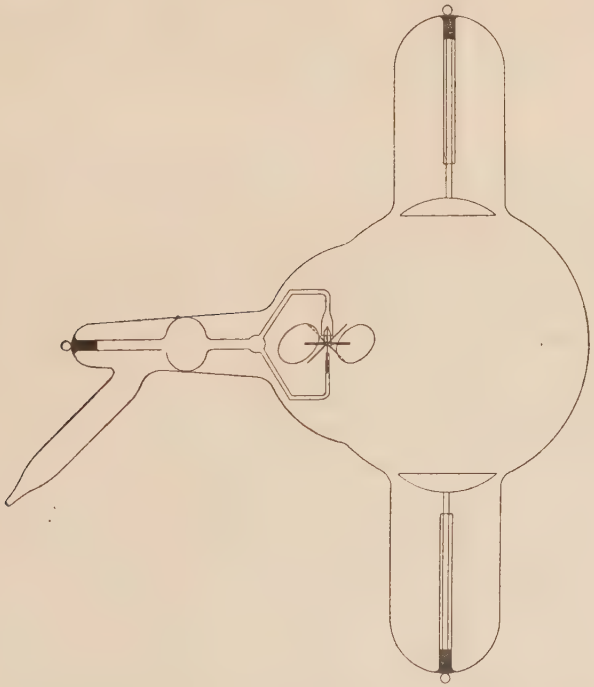
Mr. Wolff has now constructed for me a tube similar to fig. 1 of my former paper, but with the mica vanes gilded and mounted on a brass cap revolving upon a steel needle-point connected with a wire and terminal, so that the vanes can readily be earthed.

In this condition, and with the vanes so placed as to be outside the cathode-stream, it is found that this wheel behaves in a similar manner to the former non-conducting and insulated wheel. It shows a greater tendency to assume a position of stability, due evidently to electrostatic induction; but though this renders it sometimes rather troublesome in starting, still when once under weigh it will continue to rotate as long

* Read May 27, 1898.

as the tube is excited. It will occasionally, when starting, make a few reverse revolutions, due probably to electrostatic influence and momentum, and also possibly to eddy-currents in the residual gaseous matter; but it is found that when it does this it invariably reverses its rotation almost immediately, and proceeds to rotate more and more rapidly in the direction that indicates a stream of residual gaseous matter passing from the anode to the cathode. Usually, after one or two oscillations, it starts immediately to rotate in this direction.

An electrometer connected to the wheel through the pivot and needle-point shows that the vanes are always positively electrified.



In order further to investigate the matter, I have had constructed another form of radiometer-tube, as shown in the accompanying illustration. Here the vanes (of mica) are inclined, and the axis is parallel to the line joining the cathode and anode. The wheel thus rotates in a plane at right angles

to the cathode- and anode-streams ; and it is difficult to see how electrification of the vanes should in any way assist its rotation either in one direction or in another.

The wheel is arranged so that the vanes are all outside the cathode-stream, and when the tube is excited it is found invariably to rotate in a direction indicating a stream from anode to cathode. Concave aluminium cups are used for both electrodes, and the direction of rotation of the wheel is found immediately to reverse when the positive and negative connexions are transposed.

These experiments consequently confirm the hypothesis suggested in my former paper, that at very high exhaustions there exists a molecular or atomic stream from anode to cathode which carries a positive charge and travels at considerable velocity outside of the opposite cathode-stream.

Discussion.

Mr. J. Quick asked what was the minimum degree of exhaustion required to produce these results.

Prof. Boys said that the experiment gave some amount of probability to the truth of Mr. Campbell Swinton's hypothesis, but it did not altogether prove the mechanical theory of rotation to be correct. He was glad that a chance suggestion at the last discussion has led to such interesting experiments being continued.

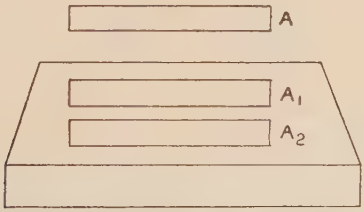
Prof. Threlfall mentioned that Boettger had devised a method for gilding mica, by a chemical process, that was much to be preferred to ordinary gilding.

Mr. Campbell Swinton said it was necessary to exhaust the tubes as completely as possible ; to a point where it was only just possible for any discharge at all to pass through them. If the rotation was due to electrification, there must still be some mechanical process whereby the charges get to the vanes—a stream of residual gas satisfied that condition.

XXI. *On a Method of viewing Newton's Rings.* *By* T. C. Porter*.

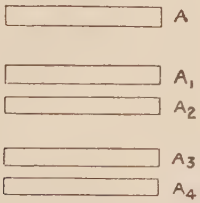
If rays of light (here supposed parallel to each other) pass through a rectangular slit A (fig. 1) and fall upon a piece of plate-glass of the same thickness as the width of the slit or greater, and if we observe the reflexion of the slit, it appears thus :—

Fig. 1.



A_1, A_2 being the first reflexions of A in the upper and lower surfaces of the glass. If the glass plate be viewed more obliquely, other reflexions, which for the present we shall neglect, will appear, all of them lying below A. If a second glass plate be added below the first, but separated from it by an interval, two more images of A will be seen below A_1, A_2, caused by the reflexion of A in the upper and lower surfaces of this second plate, A_3, A_4 in fig. 2. If now the lower glass plate be moved up till its upper surface is in contact with the

Fig. 2.

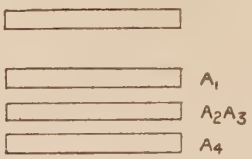


lower surface of the upper plate, the two reflexions A_3, A_4 will be seen to move up with it till, when the two plates are in contact, A_3 coincides with A_2, and the appearance of the

* Read April 22, 1898.

images is that represented in fig. 3. It is evident that the middle image is caused solely by the light reflected from the lower surface of the upper plate, and from the upper surface of the lower plate. If we substitute for the two glass plates the apparatus generally used for exhibiting Newton's rings, we can in this simple way view the rings by light coming from the two interior surfaces only, and thus completely free

Fig. 3.

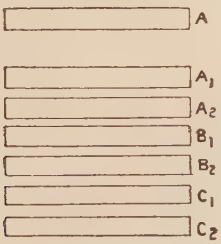


from either light reflected from the upper surface of the upper plate, or from the lower surface of the lower plate.

Thus viewed, the central area appears of a velvety black and the colours of the rings exceedingly brilliant. The whole experiment can be easily projected, and the difference in the appearance of the rings on the screen with and without the slit is very striking. But the interest of the method does not end here; for besides affording an easy and obvious proof that the rings are caused by two reflexions, one at each of the two inner surfaces, and, *under these circumstances*, by these two reflexions *only*, it also supplies a method of seeing and distinguishing the interference-curves caused by light which has undergone 1, 2, or 3 reflexions (forming the ring-system usually seen, and named after Sir Isaac), and the curves formed by the interference of rays which have suffered 4, 5, and 6 reflexions or more. For if the reflexion of the slit in a single glass plate be viewed more obliquely, as suggested before, the images are arranged as in fig. 4, where B_1 is caused by light which, originally reflected from the lower surface of the plate, has undergone a second reflexion from the upper interior surface, and a third from the lower interior surface of the plate. Similarly, the light by which B is seen has undergone five internal reflexions; C has undergone seven; C_1 nine; and so on. Now when the second plate is added beneath the first it gives rise to a similar series, but more complicated

from the fact that the second plate's reflexions are not only caused by its internal surfaces but also by its external upper surface.

Fig. 4.



The course of the rays forming the first few reflexions is easily seen from fig. 5. By backing the plate in the usual way we can practically suppress the reflexions from the lower internal surface of the lower plate; and since A_1 is the sole reflexion from the upper surface of the upper plate, it follows

Fig. 5.

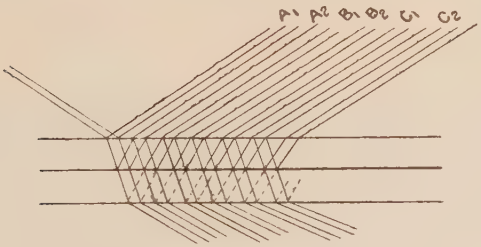


Fig. 5 *a*.

that we see in A_2, B_1, B_2, &c., the results of reflexions in (*a*) the internal upper surface of the upper plate, (*b*) the internal lower surface of the upper plate, (*c*) the external upper surface of the lower plate, and possibly (*d*) the external lower surface of the upper plate (*vide* fig. 8).

Of (*a*), (*b*), and (*c*), the images A_2, B_1, B_2, C_1, C_2 contain rays which have undergone reflexion at these respective sur-

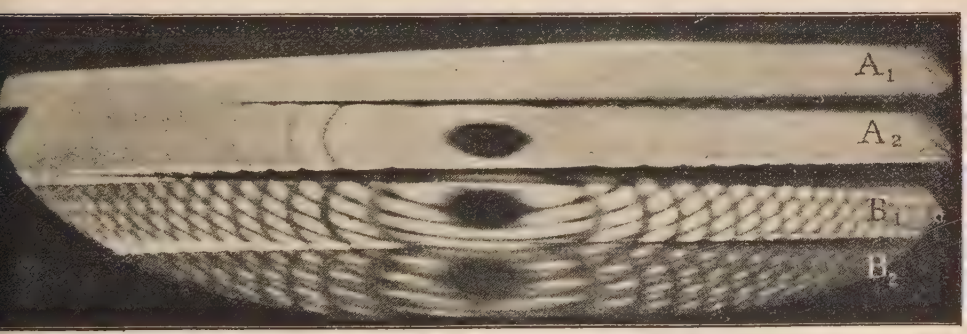
faces the number of times under the corresponding letter in the subjoined table.

	(*a*)	(*b*)	(*c*)
A_2	0	1	1
B_1	1	2	2
B_2	2	3	3
C_1	3	4	4
C_2	4	5	5

Now since the phase of a light-wave loses half a wave-length in the act of being reflected in a denser medium, and since the sum of the three rows is odd for each strip such as B_1, the light which forms the figures seen in each image of the slit will in every case lose an integral number of wave-lengths: it follows that if the centre of the primary rings is black or coloured, so far as this consideration is concerned, the centres of the secondary and tertiary rings will also be black or coloured. B_1, B_2, C_1, C_2 . . . . will therefore each contain a reflexion of the primary rings, growing weaker and weaker in intensity, not only from the loss of light at each reflexion but also from absorption, and perhaps from the scattering of a very small fraction of the light by solid particles of dust or air-bubbles, which are wont to occur even in the clearest glass. The effect of the curvature of the lower surface of the upper plate will be to displace these repetitions of the primary rings a little downwards. It is clear that any rays which once completely interfere are cut out once for all; and it follows that any interference-curves which appear for the first time in any particular image of the slit must be the result of light which has hitherto escaped interference: *e. g.* if we look at the image B_1, besides the reflexion of the primaries there is a series of rings which are exact continuations of the primaries; and, moreover, *these continuations can be traced*, though the observation is not an easy one, *right across the reflexion of the black central spot which occupies the centre of the primary rings*. If this be not due to light scattered in the body of the glass itself, it must prove that the interference which causes the black spot at the primaries is not complete, though very nearly so. In making this observation the eye must be screened from all light except that which comes from the black spot in B; and it must be borne in mind that

care is necessary to avoid the smallest particles of dust between the two plates. If there are any of these, they cause white specks in the black area, *and in its reflexion in* B, and will obviously make it easy for the eye to follow the black continuations of the primaries across the reflexion of the spot. The writer is of opinion that the interference is not complete. It seems worth notice that when the plates are very clean the darkest area of the black spot has a *sharply defined* edge, recalling the character of the black film of a soap-bubble. If there is a species of welding together of the molecules of glass at the black spot, with molecules of the gases of air mechanically entangled, the blackness would be explained, since the light can pass through without leaving the glass medium; and, on the other hand, the entangled air being, as it were, pulverized, would reflect irregularly. The fact that the edge of the spot is very sharply defined shows that it is not only the effect of the difference of wave-length. As the black spot is approached in the plane of the rings, the light *does* fall off gradually up to a certain point, at which the shade is a dark grey, but then passes "*per saltum*," to the velvety black before alluded to.

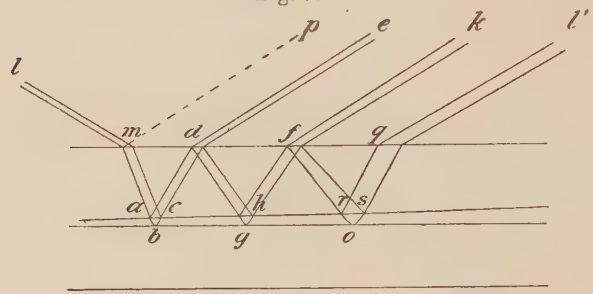
Fig. 6.



To explain how it is the continuations of the primaries are seen in B (*vide* fig. 6), see fig. 7, where *l m* is one of the rays of white light coming through the slit, reflected at the uppermost surface at *m*, giving the image A, then transmitted through the upper plate to *a* and accompanied by light reflected from the upper surface of the lower plate, which

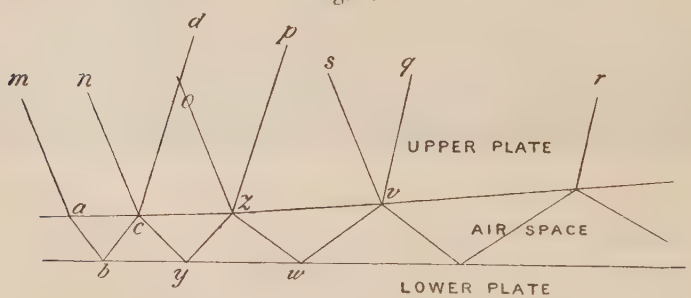
has not interfered with it. At a a part of the ray is reflected; the rest is refracted into the air-space, undergoes reflexion at b, another refraction at c; and is then transmitted to d

Fig. 7.



unless destroyed by interference: there, part is refracted into the air and forms one of the bright primaries; but a considerable part is reflected to h and g, the two reflected rays there interfering for the first time, and generating the continuations of the primary rings; for it is evident that light reflected from d might, so far as its effect at k is concerned, have been originally incident at d, in which case it certainly would have generated the continuations of the parts of the primary rings observed in A. Thus B will show (1) a reflexion of the parts of the primaries seen in A, and (2) continuations of the primaries. It is now perfectly obvious that the image B_2 (fig. 5) ought to exhibit:—(1) Faint continuations of the

Fig. 8.

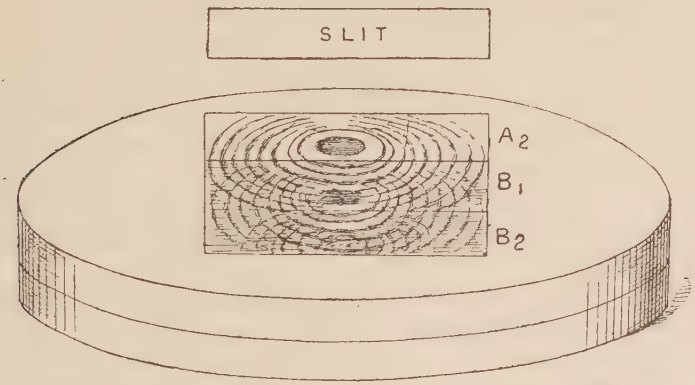


primary rings, produced by light which has escaped interference in c and h (fig. 8), but has interfered for the first

time at s, and has therefore the same effect as if it had been originally incident at f. (2) *Continuations* of the first reflexion of the primaries visible in B_1. These continuations are caused by light which has escaped interference as far as o, and is the result of the internal reflexion at d, i. e. the rings occupy the same position as if the light had been originally incident at d. (3) A new reflexion of the primary rings, first caused at c, and reflected at d, g, f, s.

Observation shows these three sets of rings in B_2 (*vide* figures 6 and 9, fig. 9 being added to make plain the arrangement by which fig. 6 was obtained), and no more. In the same way we can predict exactly what the interference systems will be like in C_1, C_2, &c. We have not yet made any attempt to examine the effect of the reflexions classed under (*d*), p. 161. If we look at fig. 8, which is part of fig. 7 on a larger scale, we see that reflexions take place at c, y, z, w, v: now if the path $a\ b\ c$ causes the part of the ray $m\,a$ which takes it to be an odd number of half wave-lengths behind that part of $n\,c$ which is reflected at c in the direction $c\,d$, and thus causes interference in $c\,d$; then

Fig. 9.



$c\,y\,z$ will cause the ray $a\,b\,c\,y\,z$ which is reflected at c to be very nearly an even number of half wave-lengths behind the incident ray $o\,z$, and therefore this ray will increase the brightness of $z\,p$: similarly in $v\,q$ the residual light which reaches v by the path $a\,b\,c\,y\,z\,w\,v$ must be an odd number of half wave-

lengths behind the light in $s\,v$, and will therefore interfere: but since in every case the diminution in intensity caused by these reflexions at $a\,b\,c\,y\,z$ must be rapid, it is practically only the reflexions $a\,b\,c\,y\,z$ that can result in visible phenomena. If the region from a to v produces, on the whole, one of the dark rings (using monochromatic light), then if $z\,p$, regarded as part of $n\,c\,y\,z\,p$ and of $o\,z$, be the darkest part of the ring, where interference is a maximum, nevertheless at z the part of the ray from a, namely $a\,b\,c\,y\,z$, will reinforce the reflected part of the ray $o\,z\,p$, and make the interference less complete than it would otherwise have been; and, similarly, a bright ring will not be quite so bright as it would be if there were no reflexions in the air-space. In short, the last-considered reflexions superimpose upon the primaries another set not in general coincident with them, and therefore in general weakening their intensity.

It is also evident, since the more refrangible the light employed, the smaller the diameters of corresponding rings, if we use white light for the generation of the rings and the subordinate systems, we can predict the result: for colour will be visible in any of the images of the slit B_1, B_2, C_1, C_2, &c. at the points of intersection of the dark curves when it has faded elsewhere. Hence co-major-axial hyperbolic broken lines of colour must result, since the various sets of curves in monochromatic light are intersecting systems of circles whose centres lie in one straight line. As the systems considered lie more and more remote from the primaries, the first hyperbola formed will be intersected by others, their numbers ever increasing, till, finally, the whole space considered is so full of them that it seems everywhere pervaded by faint white light.

The results obtained by the experiments described in this paper may be briefly stated as follows:—

(1) The method gives a very simple method of viewing Newton's rings by the light emitted from the two interior surfaces of the glass plates, free from all other light, except only that due to reflexions *in the air-space*.

(2) It reveals to the eye (for the first time) the subordinate interference systems which coexist with the primary

rings, and demonstrates which of these reflexions must be taken into account in framing the theory of the rings as they are generally viewed.

(3) It supplies a method of analysing these systems experimentally.

(4) It shows that most probably the interference of monochromatic light in forming the rings is never absolutely complete, though very nearly so.

NOTE.—The photograph (fig. 6) of the interference systems was taken by the light of sodium chloride, volatilized in a Bunsen flame. The exposures being for A_2, B_1, and B_2, 10 min., 60 min., and 240 min. respectively. The thickening of some of the lines at certain points in B_1 and B_2 is due to the faint blue light of the Bunsen burner. It is most noticeable in the immediate neighbourhood of the central spot. The plates used were Edwards's isochromatic medium, stop $f/8$. The illustration is an enlargement from $1\frac{1}{2}$ in. $\times \frac{1}{2}$ in.

Eton College, Windsor.
March 11th, 1898.

DISCUSSION.

Prof. HERSCHEL said it was rather difficult to follow the arguments of the author without witnessing the phenomena. Much complication was introduced by the successive reflexions ; it was not clear what became of them. There was no doubt as to the advantage of a narrow slit for the illumination. He thought some of the secondary reflexions might be got rid of by using plates that were slightly prismatic.

Prof. THOMPSON had, in his own laboratory, verified the advantages of the author's method of illumination. The result was a very sharply-defined first system of rings. Curves of subordinate interference were easily to be observed by this arrangement.

Prof. BOYS noticed in the photograph of the ring-systems that the independent systems of bands were distorted at the points of intersection. The intersecting curves formed a sort of honeycomb, or hexagonal system, instead of a system of curvilinear quadrilaterals. This distortion reminded him of similar effects observed in the photographs of "ripples."

Mr. Edser said he had often noticed similar distortions, but he had always been able satisfactorily to explain them as being the result of imperfect focussing. The author had referred to the fact that a thin film when viewed by reflected light appears black. A phase-change of half a wave-length takes place either on reflexion at a rarer or at a denser medium; but there is no information from which to decide between these two alternatives. The truth of the assumption that the phase-change occurs at the denser medium seems to depend, so far as experimental evidence is concerned, upon the observation that in Lloyd's bands the central one is black. To produce the Lloyd's bands only one mirror is used; the bands produced by Fresnel required three mirrors. Wernicke performed an interesting series of experiments, in which white-light reflected for various angles of incidence from a thin sheet of glass was examined spectroscopically. The spectrum was crossed by numerous black bands, and from the position of these bands in the spectrum the thickness of the glass was calculated. The calculated thickness when the angle of incidence was great differed from that obtained with small angles of incidence; the conclusion was that when light is internally reflected, even at an angle of incidence less than the angle of total reflexion, a phase-change is produced. If the space between the two plates in Mr. Porter's experiment were filled with a substance of higher refractive index than glass, a confirmation, or otherwise, of this result might be obtained.

XXII. *Galvanometers.—Third Paper.*

By Prof. W. E. Ayrton, *F.R.S.*, *and* T. Mather *.

Section A.

Introductory.

Under the above title the authors, in conjunction with Dr. Sumpner, read a paper before the Physical Society in January 1890 †, and gave a list of sensibility-records relating to a large number of instruments of various types. To facilitate comparison between different instruments, the results were reduced to a common standard as regards scale-distance, length of a scale-division, periodic time of vibration, and resistance, the values chosen for this purpose being, respectively, a scale-distance equal to 2000 scale-divisions, periodic time 10 seconds, and resistance 1 ohm. The sensibilities for steady currents were expressed as the number of scale-divisions deflexion a current of one microampere would produce (assuming proportionality) under the standard conditions, and the sensibilities for quantity as the swing produced by the instantaneous discharge of one microcoulomb when the instruments were used ballistically. The resulting numbers give a measure of the relative sensitiveness of the various instruments when quickness of action is all important. For cases where time is of little consequence, and constancy of zero (as depending on the controlling forces) is of prime importance, the results were modified by appropriately introducing the moment of inertia of the suspended systems.

Our reasons for choosing a scale-distance of 2000 scale-divisions and a period of 10 seconds as standard conditions, were, that these conditions represented more nearly than any other round numbers those under which sensitive galvanometers were generally used. For example, scales of half-

* Read May 13, 1898.

† Proceedings of the Physical Society of London, vol. x. p. 393; Phil. Mag. vol. xxx. p. 58.

millimetres used at about a metre distance were frequently met with, whilst scales of fortieths of an inch used at 3 to 4 feet distance were quite common. Further, nearly all the reflecting-instruments at the Central Technical College had long scales with divisions about $\frac{1}{24}$ of an inch placed at about 6 feet from the mirror. Since 1890, however, scales divided into single millimetres have come into much more general use, and may now be regarded as the standard type in physical laboratories. As the present tendency in galvanometer construction is to reduce the size of the needles and mirrors, and therefore the available light, it is probable that scale-distances will decrease rather than increase. We are therefore led to believe that a scale-distance equal to 1000 scale-divisions (rather than 2000) would be, on the whole, a more convenient standard to adopt. Another reason for proposing the change is to facilitate the international acceptance of a common system; for we learn from Prof. H. du Bois that our system thus modified is to be recommended for adoption at the next meeting of the Society of German Physicists and Physicians in September next at Dusseldorf. Doubtless some instrument-makers will object to the change; others, we are sure, will welcome it, for some have used the proposed standard themselves for some time. *Millimetres at a metre*, as the deflexion per microampere, is a convenient way of stating the "figure of merit" of sensitive galvanometers. Moreover, the conversion into circular measure is quite simple, 1 division at 1000 divisions distance corresponding with a movement of the suspended system through $\frac{1}{2000}$ of a radian. We would therefore suggest that in future all sensibilities be expressed in terms of this standard. In the new sensibility-records given in Table II. of the present paper we have adopted this standard; and to facilitate comparison of these records with those given in our 1890 paper, we have appropriately reduced and corrected the numbers given in the principal table of the latter paper, and embodied them in Table I.

As regards the standard periodic time, 10 seconds represented the average conditions under which sensitive galvanometers were used much more nearly than any other simple round number; hence we decided in 1890 to use this time as

the standard period to which all sensibilities were reduced. So far as we can see, this is still the most convenient standard time, and it has therefore been retained in the present paper.

The reduction to a standard resistance of one ohm has proved of considerable value in comparing various instruments, and has been frequently used by others interested in the subject. The question as to whether the "square-root law," *i. e.*

$$\text{Deflexion per microampere} \propto \sqrt{\text{resistance}},$$

or the two-fifths power law,

$$\text{Deflexion per microampere} \propto (\text{resistance})^{\frac{2}{5}},$$

should be used in this reduction is still perhaps debatable. On the whole, we consider the two-fifths law more nearly true over wide ranges of resistance; nevertheless we have given the principal results reduced according to both systems.

Recent Improvements.

Since the above-mentioned paper was published considerable advance has been made in the construction of sensitive galvanometers, more especially in the direction of reducing the dimensions of the suspended parts. We therefore thought it desirable to prepare a supplementary list of records, giving the results obtained on more recent instruments. In passing, we may remark that German and American instruments of the suspended-magnet type show the greatest progress in the reduction of dimensions and consequent quickness of action, whilst the d'Arsonval, or moving-coil type, seems to have received its greatest development in England.

To make the present lists more complete than the original one, we have inserted three additional columns, giving respectively the period of vibration of the suspended system when tested, the logarithmic decrement, and the actual sensitiveness of the instruments obtained in the test, reduced to standard scale-distance and scale-divisions. From the

first two of these columns (Columns 1 and 2 of Table II., pp. 176, 177) the relative suitabilities of the various instruments for ballistic or deflexional observation can be seen; whilst the third column (Column 3, Table II.) shows the actual "figures of merit" of the galvanometers (reduced to standard scale-distance) in the condition under which they were tested. This latter column is of considerable importance in the case of instruments whose controlling couple cannot be readily altered—for example, in the case of ordinary moving-coil instruments.

In addition to the three columns above referred to, we have also given in Table V. (p. 186) a list of records relating to moving-coil galvanometers used as voltmeters.

The values of $D/r^{\frac{1}{2}}$ or $D/r^{\frac{2}{5}}$ in columns 9 and 10 of Tables I. and II. express the relative sensibilities of the various instruments for constant period and constant resistance of coil, when the galvanometers are used as ammeters. The same columns give their relative merits as voltmeters, provided the resistance of the coil is equal to the resistance between the terminals, *i. e.* when the resistance of the connexions is small compared with that of the coil. This may be taken to be the case in instruments with fixed coils; but for moving-coil galvanometers it is by no means true, especially when the coils have low resistance, for in some instruments the resistance of the suspensions is greater than that of the coil. Further, as instruments of the d'Arsonval type cannot always have their period, and therefore their sensitiveness, easily varied, no reduction to constant period or constant resistance has been made in Table V., and the numbers in column 3 (F′) Table V. give the actual sensibilities of the respective instruments as voltmeters when the scale-distances equal 1000 scale-divisions.

We may again point out (as in our original paper *) that the comparison of galvanometers by reduction to constant period, scale-distance, and resistance, is a purely electromagnetic comparison, and takes no account of optical magni-

* Phil. Mag. vol. xxx. pp. 83–84; and Proc. Phys. Soc. vol. x. p. 420.

fication, which in some instruments is of a much more perfect nature than in others. It is also to be noticed that Table II. contains some records of galvanometers having very short periods, *e. g.* instruments numbered 4, 25, 41, 42, 43, 46, and 47; and the results of the tests on these instruments have, for the sake of uniformity, been reduced to the same period, scale-distance, and resistance, as the galvanometers having longer periods. It must not, however, be supposed that these short-period instruments could easily have their periodic time lengthened to 10 seconds—indeed, in some cases this would be practically impossible. But although the values given in columns 5, 6, 7, 12, 13, 14, 17, and 18 for the above-mentioned short-period instruments cannot as yet be realized in practice, the ratios of these numbers to those in the corresponding column for any other instrument give approximately the relative sensibilities of the two instruments compared, when both have the same *short* periodic time. To make our meaning clearer we will compare instruments numbered 24 and 43 in Table II. Referring to column 13 we find the values of $D/r^{\frac{3}{2}}$ for these instruments to be 254 and 985 respectively, indicating that No. 43 has a "factor of merit" between three and four times as large as No. 24. This must *not* be taken to mean that No. 43 is *actually* more sensitive than No. 24, but that No. 43 *would be* more than three times as sensitive as No. 24 if the period of No. 24 was reduced to, say, $\frac{1}{10}$ of a second by increasing its controlling field, and the period of No. 43 increased to $\frac{1}{10}$ of a second.

TABLE I.

1890 List of Sensitive Galvanometer Records (reduced to present standard).

	Type.	Description.	1. T.	2. λ.	3. F.	4. r.	5. M.	6. D.	7. S.*	8. V.	9. I.	10. DI.	11. $SI^{\frac{1}{2}}$	12. $\frac{D}{r^{\frac{1}{2}}}$	13. $\frac{D}{r^{\frac{2}{5}}}$	14. $\frac{S}{r^{\frac{2}{5}}}$	15. $\frac{DI}{r^{\frac{2}{5}}}$	16. $\frac{SI^{\frac{1}{2}}}{r^{\frac{2}{5}}}$	17. $\frac{D}{r^{\frac{2}{5}}V}$	18. $\frac{S}{r^{\frac{2}{5}}V}$	19. $\frac{DI}{r^{\frac{2}{5}}V}$	20. $\frac{SI^{\frac{1}{2}}}{r^{\frac{2}{5}}V}$
1.	Thomson. One pair of coils.	Elliott's Tripod (F.T.C.) ‡	3·62	0·19	0·50	0·6	3·8	3·8	2·38	...	...	...	...	5	4·7	2·95							
2.		Tripod (F.T.C.) ‡	2·55	0·33	6·65	50	102	102	64	...	0·04	4·08	12·8	14·4	21·4	13·5	0·85	2·7					
3.		Elliott's Tripod, specially light needles †	20	...	30,800	3,860	7,550	7,700	4,850	125	0·01	77	485	124	283	178	2·83	17·8	2·27	1·43	0·023	0·15	
4.		„ „ „ „ „ †	20	...	64,000	6,000	15,750	16,000	10,000	125	0·01	160	1,000	206·5	493	308	4·93	30·8	3·95	2·47	0·040	0·25	
5.	Thomson. Two pairs of coils.	Elliott's Glass Cylinder (F.T.C.) ‡	2·33	0·26	66	4,600	1,220	1,220	760	...	...	...	...	18	41·5	26							
6.		Latimer Clark & Muirhead (F.T.C.) ‡	1·33	0·20	95	20,000	8,700	5,350	3,360	...	...	...	...	38	101·5	64							
7.		Elliott's Stock Pattern †	20	...	14,200	7,000	4,300	3,550	2,225	30	0·013	46	254	42·5	102·5	64·5	1·34	7·35	3·4	2·15	0·045	0·025	
8.		„ Large Coil Pattern †	20	...	30,400	12,300	7,680	7,600	4,775	240	0·013	99	545	68·5	175	110	2·27	12·6	0·73	0·46	0·010	0·05	
9.		„ „ „ „	20	...	43,600	30,000	10,710	10,900	6,850	240	0·013	142	780	63	176	110	2·3	12·7	0·74	0·46	0·010	0·05	
10.		„ Special Form †	20	...	80,000	30,000	20,000	20,000	12,500	250	...	...	...	115·5	324	202·5	...	...	1·34	0·30			
11.		„ Ayrton & Perry Ballistic (Japan)	39·5	...	11,200	20,000	1,070	715	450	...	...	...	...	5·25	13·65	8·55							
12.		Langley's Bolometer	40	...	2,000	20	125	125	78	...	...	...	...	28	38	23·5							
13.	Mudford's Thomson. One pair of coils.	Simple Reflecting, Fig. 5 (C.T.C.)	5·8	...	59	26	347	175	110	72	0·042	7	22·5	33·3	47·2	30	1·96	6·1	0·66	0·42	0·028	0·08	
14.		„ „ „ „	5·7	...	350	2,457	1,375	1,080	680	72	0·042	45·5	142	21·8	47·8	30	2·00	6·15	0·67	0·42	0·028	0·08	
15.		„ „ „ „	5·7	...	547	6,410	2,720	1,680	1,055	72	0·042	71	218	21	50·5	31·5	2·13	6·5	0·70	0·44	0·030	0·085	
16.		„ „ „ Nalder Bros.†	20	...	3,440	2,086	880	860	540	72	...	...	...	18·8	40·5	20·5	...	...	0·56	0·36			
17.		Bell Magnet Ballistic, Nalder Bros.†	20	...	2,700	7,800	720	675	425	...	...	...	...	7·65	18·7	11·8							
18.		„ „ „ „ „ †	20	...	2,400	10,000	640	600	378	...	...	...	...	6·0	15·1	9·5							
19.		Simple Reflecting, Fig. 5 (C.T.C.)	5·7																				

21.	Mudford, Thomson. pairs of coils	„ „ (C.T.C.) [(C.T.C.)	7·9	...	300	686	1,660	300	305	88	0·084	67	147	30·5	59	37	4·91	10·8	0·67	0·42	0·055	0·12		
22.		„ Variable damping, Large coils	10·0	...	2,140	9,744	4,200	2,140	1,320	200	0·052	111	301	21·8	54	33·5	2·81	7·65	0·27	0·17	0·014	0·038		
23.		High Insulation Galv., Fig. 7 (C.T.C.)	10·0	...	27,100	360,000	50,000	27,100	17,000	236	0·01	271	1,700	45	162	101	1·62	10·1	0·69	0·43	0·007	0·043		
24.		„ „ „ Nalder Bros.†	20	...	62,800	350,000	15,700	15,700	9,850	...	...	...	...	26·6	95	59·5								
25.	Rayleigh, Gray, or Rosenthal.	Rosenthal Non-Astatic, Fig. 8 (C.T.C.)	9	0·13	10·4	44	22·9	12·7	8	0·29	1·5	19	9·8	1·91	2·8	1·75	4·2	2·15	9·65	6·05	14·5	7·4		
26.		„ Astatic, Fig. 10 (C.T.C.)	10·5	...	15·3	44	25	13·9	8·7	0·29	0·7	9·7	7·25	2·1	3·05	1·9	2·14	1·6	10·5	6·55	7·4	5·5		
27.		„ Non-reflecting „	8	...	2·9	5·2	...	4·53	2·85	2·0	0·21	0·96	1·31	2·0	2·35	1·47	0·49	0·68	1·18	0·74	0·25	0·34		
28.		Gray's Galv. (Pr. Roy. Soc. 1884, p. 287)	22	...	155	30,000	32	32	20	51	50	1600	141·5	0·19	0·52	0·3	26	2·3	0·01	0·006	0·5	0·45		
29.		Ballistic „ Figs. 11 & 12 (C.T.C.)	14·4	0·035	65	4,090	54	31·5	19·6	36	17	535	82·5	0·5	1·13	0·7	19·5	3·0	0·03	0·02	0·55	0·085		
30.		Threlfall's Galv. (Phil. Mag. Dec. 1889, p. 460)	80	...	120	15,680	2·90	1·87	1·18	72	...	...	...	0·015	0·039	0·025	...	...	0·00054	0·00034				
31.	d'Arsonval.	Jolin's High-resistance (C.T.C.)	1·75	...	3·98	750	137	130	81·5	2·8	30	3900	448	4·75	9·2	5·75	276	31·5	3·3	2·05	99	11		
32.		Carpentier's Ordinary	0·8	...	1·34	208	286	188	118	...	...	...	...	13	22·2	14								
33.		Jolin's Low-resistance (C.T.C.)	1·5	...	0·475	3·8	39	21	13·2	0·6	5·5	115	31	10·8	12·5	7·75	67	18·2	21	13	113	30·5		
34.		Invariable Sensibility, Large type (C.T.C.)	2·6	...	6·1	21	190	90·5	57	0·96	13	1180	205	19·8	27	16·8	350	61	28	17·5	365	63		
35.		Nalder's type fitted with invariable sensibility device — Pole pieces in (C.T.C.)	0·63	...	0·335	274	145	84·5	53	0·2	1·5	127	64·5	5·1	8·9	5·6	13·5	6·85	44·3	28	67·5	34·3		
36.		Nalder's type fitted with invariable sensibility device — „ out „	1·55	...	2·91	274	209	121	76	0·2	...	...	...	7·3	12·8	8·1								
37.		Carpentier's Milliammeter (C.T.C.)	...	...	...	183	...	36·5	22·9	3·2	25	910	114	2·7	4·5	2·8	113	14	1·4	0·87	35	4·35		
38.	Flat coil or Multiplier type.	Single-needle Post-Office pattern (C.T.C.)	2·5	...	5·95	800	...	95	60	...	...	...	...	3·35	6·55	4·1								
39.			...	...	...	400	...	151·5	95	22	0·2	30·3	42·5	7·55	13·8	8·7	2·75	3·88	0·63	0·40	0·125	0·175		
40.		Suspended needle instrument with	4·0	...	11·6	350	...	73	45·9	22	0·2	14·6	20·5	3·9	7·0	4·4	1·41	1·97	0·32	0·20	0·065	0·09		
41.		Coils 6 centimetres long outside	4·0	...	10	64	...	62·5	39·4	22	0·2	12·5	17·5	7·8	11·9	7·5	2·38	3·33	0·54	0·34	0·105	0·15		
42.		„ 2·7 „ broad „	16	...	19	4·3	...	7·4	4·65	22	0·2	1·48	2·05	3·55	4·13	2·6	0·83	1·14	0·19	0·12	0·035	0·05		
43.		„ 2·2 „ high „	...	...	...	4·3	...	17·6	11·1	22	0·2	3·52	5·0	8·5	9·8	6·2	1·96	2·25	0·43	0·28	0·09	0·125		
44.		Winding 0·6 „ thick	9·6	...	3·14	0·87	...	3·4	2·14	22	0·2	0·7	0·95	3·65	3·6	2·27	0·75	1·01	0·16	0·11	0·035	0·045		
45.			...	...	...	0·87	...	8·35	5·25	22	0·2	1·65	2·35	9·0	9·0	5·55	1·75	2·5	0 41	0·25	0·075	0·115		
46.		Nalder's Detector, nearly astatic †	...	...	490	2,000																		
47.		„ „ partially „ †	...	...	42	2,000																		

* The numbers in this column are calculated from those given in column 6 without allowing for damping.
† The constants of these instruments have been kindly furnished by the makers.
‡ These instruments were tested after being in use for some time and without their needles being remagnetized.

TABLE II.

Supplementary List of Sensitive Galvanometer Records.

Number.	Type.	Description.	1. T.	2. λ.	3. F.	4. r.	5. M.	6. D.	7. S.*	8. V.	9. I.	10. DI.	11. $SI^{\frac{1}{2}}$	12. $\frac{D}{r^{\frac{1}{2}}}$	13. $\frac{D}{r^{\frac{2}{5}}}$	14. $\frac{S}{r^{\frac{2}{5}}}$	15. $\frac{DI}{r^{\frac{2}{5}}}$	16. $\frac{SI^{\frac{1}{2}}}{r^{\frac{2}{5}}}$	17. $\frac{D}{r^{\frac{2}{5}}V}$	18. $\frac{S}{r^{\frac{2}{5}}V}$	19. $\frac{DI}{r^{\frac{2}{5}}V}$	20. $\frac{SI^{\frac{1}{2}}}{r^{\frac{2}{5}}V}$
1.	Kelvin. One pair of coils.	Elliott's Tripod pattern † 1892	20	...	53	0·5	13·1	13·2	8·2	...	...	...	...	18·6	17·4	10·9						
2.		,, ,, ,, † ..... ,,	20	...	1,000	14·8	250	250	157	...	...	...	...	65	85	53·5						
3.		,, ,, ,, † ..... ,,	20	...	500	19·9	125	125	79	...	...	...	...	28	37·7	23·8						
4.		McKittrick's Oscillograph Galvanometer. (Soft-iron needle)...1896	0·000377	small	0·0014	270	296,000	985,000	620,000	16	1×10^{-6}	0·99	620	60,000	105,000	67,000	0·11	67	6,550	4,200	0·0066	4·2
5.	Kelvin. Two pairs of coils.	[1892																				
6.		Elliott's glass cylinder pattern †. Queen & Co., Philadelphia. (Re-magnetized by Wadsworth). 1894	20	...	3,990	80·5	997	998	625	...	...	...	...	111	173	108						
7.		Prof. F. W. Burstall's Galv. ...1895	20	...	10,000	309	2,500	2,500	1,570	...	...	...	...	143	253	158						
8.		Hartmann & Braun, Frankfürt †.	3	...	11	15	366	122	76	118	...	...	...	31·5	41·3	26	...	...	0·35	0·22		
9.		[1896	9	...	500	3,500	620	620	390	...	...	...	...	10·5	23·5	14·9						
10.	Mudford-Kelvin. Two pairs of coils.	Mudford Galvanometer. (Designed by Mather; Light needle)...1891	10·9	1·0	11,200	3,887	10,700	10,350	6,500	25	0·01	104	650	166	380	240	3·8	24·0	15·2	9·6	0·15	0·96
11.		W. B. Snow's Bolometer Galvanometer. (Small light needle).1892	20	...	22,200	140	16,500	5,550	3,500	155	...	...	...	470	770	485	...	...	5·0	3·2		
12.		E. L. & E. F. Nichols' Galvanometer. (Very small light needle) ...1892	10	...	6,650	9·3	9,800	6,650	4,180	...	...	...	...	2,200	2,730	1,700						
13.		E. L. Nichol's & W. S. Franklin's Galv. (Small light needle). ...1892	14	...	1,390	15	840	705	443	...	...	...	...	182	239	149						
14.		Ditto (ditto) ..... ,,	7	...	1,390	150	3,400	2,835	1,780	...	...	...	...	231	381	240						
15.		Nalder Bros. Galv. modified by Bourne. (Vertical needle). ...1893	17	...	7,800	19,640	9,700	2,700	1,700	...	0·05	135	380	19·3	52	32·5	2·6	7·25				
16.		Elliott's Galv. (Des. by Wadsworth; very small light needle).....1894	20	...	25,000	86	6,250	6,250	3,930	600	...	...	...	675	1,050	660	..	...	1·75	1·1		
17.		Nalder Bros. Galvanometer. (Light needle) † 1895	20	...	35,500	6,490	8,900	8,900	5,550	115	...	...	...	110	265	165	...	...	2·3	1·43		
18.		Weiss' Galvanometer. (Small vertical needle) 1895	10	...	...	146	...	665	417	...	...	...	...	55	90·5	56·5						
19.		Ditto. (Very small vertical needle) ..... ,,	10	...	...	...	...	...	...	...	...	...	...	600								
		Ditto (ditto) ..... ,,	10	...	...	...	...	...	...	...	...	...	...	750								

21.	Paschen-Kelvin. pairs of oval coi	Ditto (ditto) 1893	15	...	101,000	60	135,000	45,000	28,300	...	...	...	...	5,800	8,750	5,500									
22.		Above instrument after 3 years' use. (Connected up differently)..1896	17	...	48,000	15	41,700	16,700	10,500	...	...	...	...	4,310	5,650	3,550									
23.		Paschen's Galvanometer. (Very small light needle) 1896	10	...	27,000	40	27,000	27,000	17,000	...	...	...	...	4,280	6,150	3,850									
24.		Ditto, modified by Harker, Owens College. (Vertical needle) 1896	20	...	4,440	40	2,770	1,110	695	33	...	...	...	176	254	159	...	...	7·7	4·83						
25.	One oval coil.	Hotchkiss & Millis' Oscillograph Galvanometer. (Soft-iron needle).....1895	0·000256	small	0·000031	4·1	29,200	46,300	29,000	3·6	$2·2\times10^{-6}$	0·10	43	22,800	26,300	16,500	0·058	24·5	7,300	4,600	0·016	6·8			
26.		Holden d'Arsonval. (Rectangular coil) 1890	0·8	...	0·54	26·7	129	83·5	52·5	...	...	...	..	16·1	22·4	14·1									
27.		Ayrton-Mather Ballistic. (Narrow coil) C.T.C.....1892	2·2	0·05	32·5	13·2	1,100	665	418	0·7	0·07	46·5	110	183	238	149	16·5	39	340	213	23·5	55			
28.		Ayrton-Mather, well damped. (Paul's No. 6).....1892	4·0	2·6	47·5	300	500	500	314	...	...	...	...	29	51	32									
29.		Holden d'Arsonval. (Rectangular coil) 1893	1·42	1·77	0·39	2·22	26·4	25·4	15·9	...	...	...	...	17·2	18·1	11·3									
30.		Queen & Co., Philadelphia. (Large narrow-coil) 1893	aperiodic	...	200	178																			
31.		Sullivan's "Universal" Galvanometer §.....1895	4·61	...	288	1,260	5,100	1,325	830	...	...	...	...	37·3	76	47·5									
32.		Hartmann & Braun, Frankfurt†1896	aperiodic	...	285	1,600																			
33.		Crompton's Large pattern. (Circular coil) 1896	12·1	small	262	103·3	357	179	112	...	...	...	...	17·6	27·8	17·5									
34.		Ditto (do.) (Camb. Eng. Lab.) „	16·1	0·03	138	100	55	53	33·3	3·12	15·6	825	131	5·3	8·4	5·25	131	20·8	2·7	1·68	42	6·65			
35.		Ditto (do.) (W. R. Cooper, Esq.)1897	7·2	0·018	70	62·6	135	135	85	...	...	...	...	17·1	25·8	16·2									
36.		Ditto Midget pattern (do.) (Camb. Eng. Lab.) 1897	10·0	small	66·3	57·6	77	66·3	41·7	1·33	5·9	391	101	8·75	13·1	8·2	77	19·5	9·8	6·15	58	14·7			
37.		Nalder Ballistic. (Rectangular coil). (Camb. Eng. Lab.) 1897	12·9	0·04	760	1,053	428	455	286	...	...	...	...	14·0	28·5	18									
38.		Nalder Damped (do.) No. 6283.1897	3·25	0·264	3·62	29·5	3·12	34·3	21·5	...	...	...	...	6·3	8·85	5·55									
39.		Andersen's Galvanometer. (Circular coil) 1897	4·7	2·63	6·45	257	97·6	64	40·2	...	...	...	...	4·0	6·95	4·38									
40.		Ditto (do.) ..... „	4·5	1·62	3·88	32·4	48·5	31·8	20	...	...	...	...	5·55	7·9	5·0									
41.		Ayrton-Mather Speaking Galv. (A) (Narrow coil) C.T.C.1897	0·0965	...	0·55	244	72·0	5,850	3,625	0·54	0·034	199	675	375	650	408	22·1	75	1,200	755	41	139			
42.		Ditto (B) „ „ „	0·0868	...	0·52	267	8800	6,900	4,325	0·54	0·036	248	820	421	735	463	26·5	87·5	1,365	855	49	163			
43.		Ditto (C) „ „ „	0·0925	...	0·76	243	11200	8,900	5,600	0·54	0·031	276	980	570	985	623	31·4	109	1,830	1,150	53·5	207			
44.		Ayrton-Mather New pattern Galv. (Narrow coil) C.T.C.1897	5·84	0·07	230	22·2	965	680	424	...	...	...	...	145	197	124									
45.		Ditto „ „ „	7·6	...	102	1·9	250	177	111	0·6	0·30	53	56	128	137	86	41	43·3	228	142	68	72			
46.		Duddell's Oscillograph. (Very narrow coil and electromagnet). 1897	0·000358	...	0·000156	1·3	164,500	122,000	76,500	...	about 30×10^{-6}	...	...	107,000	109,000	68,500	3·3	374							
47.		Ditto „ „ 1898	0·000093	...	0·00042	2·08	1,920,000	4,780,000	3,000,000	...	6×10^{-8}	...	...	8,300,000	4,560,000	2,240,000	0·21	548							

* The numbers in this column are calculated from those given in column 6 without allowing for damping.

† The constants of these instruments have been kindly furnished by the makers.

§ The inventor of this galvanometer says its period was intentionally made long for a special purpose and that the period could be considerably reduced without lessening the deflection per microampere.

Explanation of Tables I. and II.

Column 1 (or T) gives periodic time of vibration in seconds when tested.
„ 2 (or λ) „ logarithmic decrement of motion when tested.
„ 3 (or F) „ deflexion in divisions per microampere (as tested) when scale-distance = 1000 divisions.
„ 4 (or r) „ resistance of coil in ohms.
„ 5 (or M) „ deflexion in millimetres per microampere when undamped period is 10 seconds and scale placed as in actual use of the instrument.
„ 6 (or D) „ deflexion in divisions per microampere when undamped period = 10 seconds and scale-distance = 1000 divisions.
„ 7 (or S) „ swing per microcoulomb under same conditions as in last.
„ 8 (or V) „ volume occupied by convolutions of wire, in cubic centimetres (approximately).
„ 9 (or I) „ moment of inertia of the suspended system, in C.G.S. units (approximately).
„ 10 (or DI) „ deflexion in divisions (scale-distance = 1000 divisions) per microampere for constant controlling moments, and for a periodic time equal to $10\sqrt{I}$ seconds.
„ 11 (or $SI^{\frac{1}{2}}$) „ swing per microcoulomb under same conditions as in last.

Columns 12, 13, and 14 give the deflexion per microampere and swing per microcoulomb, when the period is 10 seconds and resistance of each instrument is one ohm*.

Columns 15 and 16 give the deflexion per microampere and the swing per microcoulomb, for the same controlling moment, the resistance of each instrument being one ohm.

„ 17 „ 18 give the deflexion per microampere and the swing per microcoulomb per cubic centimetre of coil, when the period is 10 seconds and the resistance of each galvanometer one ohm.

„ 19 „ 20 give the deflexion per microampere and the swing per microcoulomb per cubic centimetre of coil, for the same controlling moment, and the resistance of each instrument equal to one ohm. Periodic time of any instrument is $10\sqrt{I}$ seconds.

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* Column 12 is based on the assumption that the sensitiveness is proportional to the square root of the resistance, whilst columns 13, 14, 15, &c. assume that sensitiveness varies as the two-fifths power of the resistance. (See Phil. Mag. July 1890, p. 85 *et seqq.*, and Proc. Phys. Soc. vol. x. p. 422.)

Comparison as Ammeters.

On comparing Table II. of the present paper with the one published in 1890 (shown in Table I., pp. 174, 175), it will be observed that in columns 12 and 13 (which express the factors of merit of the various instruments when time is of importance) the values are on the average greatly increased. For example, the highest values in the 1890 table were given by one of Messrs. Elliott Bros.' Thomson instruments of special form (see line 4, Table I.), for which $D/r^{\frac{1}{5}}$ was 206·5 and $D/r^{\frac{2}{5}}$ 493, and these numbers were far above any others in the list made out at that time. These figures are now surpassed by no less than eleven instruments given in the present list (omitting the oscillograph records in lines 4, 25, 46, and 47); and three of the eleven instruments are of the moving-coil type.

The highest values in the present table (omitting the oscillographs previously mentioned) are those given by one of Prof. Paschen's instruments, line 21, Table II., and are 5800 and 8750 for $D/r^{\frac{1}{5}}$ and $D/r^{\frac{2}{5}}$ respectively. We therefore see that Prof. Paschen has produced an instrument which, at a given periodic time and of a given resistance, is about twenty times as sensitive as the best given in our previous paper. Other instruments of the moving-magnet type giving very high factors of merit are:—

(*a*) Two Paschen instruments. (Lines 20 and 23 of table.)
(*b*) Prof. Nichols' galvanometer *. (Line 11 of table.)
(*c*) Smithsonian Institution instrument, designed by Wadsworth † and constructed by Messrs. Elliott Bros. (Line 15 of table.)
(*d*) Prof. W. B. Snow's bolometer galvanometer ‡. (Line 10 of table.)
(*e*) Two galvanometers by Weiss §. (Lines 18 and 19 of table.)

To show at a glance the character of the improvement, the

* 'The Galvanometer,' by Prof. E. L. Nichols, p. 80.
† Phil. Mag, vol. xxxviii. p. 553.
‡ Physical Review, vol. i. p. 37.
§ *C. R.* vol. cxx. p. 728, and *Journ. Phys.* vol. iv. p. 212.

numbers for these instruments are abstracted from the main table and put in order of magnitude below.

TABLE III.

Description.	Values of		
	r.	$D/r^{\frac{1}{2}}$.	$D/r^{\frac{2}{3}}$.
Paschen's Galvanometer	60	5,800	8,750
,, ,, ...	40	4,280	6,150
Nichols' ,, ...	9·3	2,200	2,730
Weiss' ,, ...	.....	750	
Wadsworth's ,, ...	86	675	1,050
Weiss' ,, ...	... ..	600	
Snow's ,, ...	140	470	770
Elliott's Galvanometer (from Table I.)	6,000	206·5	493

For the moving-coil instruments, the numbers are

Description.	r.	$D/r^{\frac{1}{2}}$.	$D/r^{\frac{2}{3}}$.
Ayrton-Mather Galvanometer	243	570	985
,, ,, ,, ...	267	421	735
,, ,, ,, ...	244	375	650
Invariable Sensibility Galvanometer (from 1890 Table)	21	19·8	27

These increases in the factors of merit have been brought about chiefly by reducing the dimensions of the suspended parts, and the use of better magnet steel has probably contributed something to the improvement.

The three moving-coil instruments which have surpassed the highest values of $D/r^{\frac{1}{2}}$ and $D/r^{\frac{2}{3}}$ given in our 1890 table are of the narrow coil type, the cross-section of the winding being

of the form described before this Society in 1890*, viz., two equal circles in contact.

To attain the results here shown (Table III.), the transverse dimensions of the coils have been made smaller than usual and very powerful permanent magnets employed. When we compare the records of these instruments with the best listed in our 1890 table (particulars of which are given in the last line of the preceding table) the improvement is evident; for whereas the highest value of $D/r^{\frac{2}{5}}$ given by a moving-coil instrument in 1890 was 27, the present highest value is 985.

It is also interesting to notice that so far as absolute sensitiveness for current is concerned (*i. e.* the "millimetres at a metre" per microampere, unreduced for period and resistance) the progress made with moving-coil instruments during the last eight years has been considerable. In our 1890 Table the d'Arsonval of greatest absolute sensitiveness as an ammeter was the "Large invariable sensibility" instrument given in line 4 of the d'Arsonval section of that list (see line 34, Table I.). This galvanometer gave 6·1 divisions per microampere, its period being 2·6 seconds, and coil-resistance 21 ohms. In the more recent list, Table II., the above record is surpassed by no less than thirteen instruments out of the twenty-one tabulated, the most sensitive d'Arsonval galvanometer to current being one made by Messrs. Nalder Bros. & Co., which gives 760 divisions per microampere, its period being 12·9 seconds and resistance 1053 ohms (see line 37, Table II.).

On comparing column 15 of Table II. with the corresponding one in the 1890 Table, which gives the values of $DI/r^{\frac{2}{5}}$, *i. e.* numbers proportional to the deflexion per microampere with *constant controlling moment* †, we observe that no increase is perceptible. On the contrary, the numbers in the columns of the present table are on the average smaller than in the previous one. This is what might be expected from a diminution of the dimensions of the moving parts of the

* "On the shape of Movable Coils," &c., Proc. Phys. Soc. vol. x. p. 376.

† See paper on "Galvanometers," Phil. Mag. July 1890, and Proc. Phys. Soc. vol. x. p. 421.

instruments. An examination of columns 16 in the two tables (values of $SI^{\frac{1}{2}}/r^{\frac{5}{2}}$) leads to a conclusion similar to that deduced from a comparison of columns 15, the only instance of increased values being in the three narrow-coil d'Arsonval instruments just referred to.

It is perhaps worth while again calling attention to the fact of the superior sensitiveness (as ballistic instruments) of galvanometers with suspended parts having small inertia, a fact which is clearly shown by a study of columns 9 and 14 of the tables.

Prof. R. Threlfall's "New" Galvanometer.

Since the paper was read our attention has been directed to a very sensitive galvanometer designed by Prof. R. Threlfall, M.A. (an account of which is given in the Phil. Trans. of the Roy. Soc. 1896, vol. clxxxvii. pp. 80–97), and used by him and Mr. J. H. D. Brearley in their "Researches on the Electrical Properties of Pure Sulphur." This instrument is in many respects a remarkable one. Unfortunately the details of the sensibility tests given in the printed paper are hardly sufficient to enable us to make the reductions to constant period and scale distance, as is done in Table II., so we give the results separately below. The tests were made by observing the throw produced on reversing a small current through the galvanometer, the throws being read by a telescope at about 267 centimetres from the galvanometer. The telescope had a micrometer eyepiece and a glass scale in it divided to fifths of a millimetre.

Date.	Period in seconds.	Current in amperes.	Throw on reversal in micrometer divisions.	Amperes per throw of 1 division.
Oct. 17th, 1892.	14·5	$2{\cdot}74\times10^{-11}$	7·9	$3{\cdot}5\times10^{-12}$*
Sept. 1893	25	$2{\cdot}74\times10^{-12}$	19·3	$1{\cdot}43\times10^{-13}$†

* Phil. Trans. vol. 187, line 1 of table vi. p. 95.

† Ibid. line 1 of table vi. p. 96.

The definition of the optical system was sufficiently good to permit the micrometer-scale being read to a fifth of a division, so that currents of one-fifth the magnitude of those given in the last column of the above Table could be detected.

Oscillographs.

Oscillographs, or instruments for showing the character of rapidly-varying currents or potential-differences, were practically unknown when our 1890 paper was read. They have become possible by the great reduction of dimensions of moving parts which has gone on continuously since that date. An oscillograph may be defined as a galvanometer of very short period. The short periodic time is usually obtained by using very small moving parts and a very strong control. In these instruments the reduction of size has been carried far beyond what has yet been attempted for ordinary scale-reading work. Mr. McKittrick's* galvanometer has a moving system whose moment of inertia is about one-millionth of a C.G.S. unit, or about one ten-thousandth the inertia of an ordinary *light* astatic needle fitted with a $\frac{3}{8}$ in. *light* mirror. The record of this instrument is shown in line 4, Table II., and line 25 gives the record of another oscillograph galvanometer by Messrs. Hotchkiss and Millis †. Both the above instruments are of the moving magnet type with soft-iron needles adopted for similar purposes by Blondel, Moler, and others. Two other oscillographs are given in lines 46 and 47 of Table II., which are of the moving-coil type, and were made by Mr. W. Duddell ‡ of the Central Technical College.

For convenience of reference the principal numbers relating to oscillographs are tabulated separately below, reductions being made to a constant period of a thousandth of a second, instead of 10 seconds, and the deflexions stated in terms of an ampere, instead of a microampere.

* Trans. A. I. E. E. vol. xiii. Nos. 6 and 7.

† Phys. Review, vol. iii. p. 49. In Phys. Rev. vol. iv. p. 128, Mr. Millis gives an account of another instrument of very short period. Unfortunately the data are insufficient to permit of comparison with other oscillographs.

‡ See "Oscillographs," Electrician, Sept. 10, 1897, p. 638.

Table IV.

Description.	t.	f.	r.	I approximately.	$d/r^{\frac{1}{2}}$.	$d/r^{\frac{2}{5}}$.
Hotchkiss and Millis' galv., 1895	0·256	31	4·1	$2{\cdot}2\times10^{-6}$	228	263
McKittrick's galv., 1896	0·377	1400	270	$1{\cdot}0\times10^{-6}$	600	1050
Duddell's galv., 1897 .	0·358	156	1·3	30×10^{-6}	1070	1090
„ „ 1898 .	0·093	420	2·08	6×10^{-8}	33,000	35,600

Explanation of Table IV.

Column t gives the periodic time in thousandths of a second when tested.

„ f „ deflexion in divisions per ampere as tested when scale-distance = 1000 scale-divisions.

„ r „ resistance of instrument in ohms.

„ I „ moment of inertia of the moving systems in C.G.S. units approximately.

Columns $d/r^{\frac{1}{2}}$ and $d/r^{\frac{2}{5}}$ give the deflexions per ampere when the periodic time is one thousandth of a second and the resistance of each instrument is one ohm. Column $d/r^{\frac{1}{2}}$ is based on the assumption that the sensitiveness varies as the square root of the resistance, whilst column $d/r^{\frac{2}{5}}$ assumes that sensitiveness is proportional to the two-fifths power of the resistance.

On comparing Mr. Duddell's 1897 instrument with those of Messrs. Hotchkiss and Millis and Mr. McKittrick, we find its factor of merit, as indicated by $d/r^{\frac{1}{2}}$ (the expression most suited to instruments having comparatively low resistance), is decidedly greater than either of the other two, in spite of the fact of its wires being of phosphor-bronze instead of copper. It may also be pointed out that the moment of inertia of the moving parts of Mr. Duddell's 1897 oscillograph is very considerably greater than the inertias of the two first-mentioned instruments, being in the approximate ratios of 30 : 2·2 : 1, thus showing the great superiority of the moving-coil type for work of this nature, where short periodic time (or high frequency) and good sensitiveness are necessary. In his latest oscillograph Mr. Duddell has carried the reduction of dimensions much further than has previously been attempted,

and has produced an instrument whose periodic time is less than a ten-thousandth of a second (frequency 10,680). Although its resistance is little more than 2 ohms, it actually gives 420 divisions per ampere at standard scale-distance, and its factors of merit, as represented by $d/r^{\frac{1}{2}}$ and $d/r^{\frac{2}{5}}$, are more than 30 times those of any instrument made previously.

Addition made October 1898.

[We desire to mention that all the oscillographs referred to in this subsection are modifications of forms first described by Mons. A. Blondel, *Comptes Rendus*, 1892, vol. cxvi. pp. 502 & 748, where both the principles and theory of Oscillography are fully dealt with. M. Blondel has kindly furnished us with the following approximate particulars relating to one of his moving magnet instruments made for workshop use :—

Periodic time of vibration (undamped) $\frac{1}{600}$ second.

Resistance 7 ohms.

Deflexion per ampere at 1 metre = 40 millim.

Dimensions of needle { Iron $7 \times 1 \times 0{\cdot}2$ millim. / Mirror 7×1 to $1{\cdot}5 \times 0{\cdot}5$ „

It should be pointed out that this particular oscillograph was designed with a view of obtaining plenty of light, so that the curves could be shown on a screen by aid of a glow-lamp, rather than for large sensitiveness.

Expressing the results in the form adopted in Table IV. we get the following values :—

t.	f.	r.	I (approximately).	$d/r^{\frac{1}{2}}$	$d/r^{\frac{2}{5}}$
0·16	40	7	25×10^{-6}	540	660

the meanings of t, f, r, I, and d being as mentioned in the "Explanation of Table IV."]

d'Arsonval Galvanometers as Voltmeters.

In Table V. the actual sensibilities of various d'Arsonval galvanometers when used as voltmeters are given, reductions having been made to bring them to the same scale-distance and scale-divisions only.

TABLE V.—List of d'Arsonval Galvanometer Records (as Voltmeters).

Number on list.	Description.		1	2	3	4	5
			T.	λ.	F′.	r.	r'.
26	Holden d'Arsonval (Rectangular coil)	1890	0·8	...	0·020	26·7	27·2
27	Ayrton-Mather Ballistic (Narrow coil) C.T.C.	1892	2·2	0·05	1·31	13·2	24·8
28	„ Damped (Paul's No. 6)	1892	4·0	2·6	0·158	300	304
29	Holden d'Arsonval (Rectangular coil)	1893	1·42	1·77	0·136	2·22	2·86
30	Queen & Co., Philadelphia (Large narrow coil)	1893	Aperiodic	...	1·13	...	178
31	Sullivan's "Universal" Galvanometer *	1895	4·61	...	0·224	...	1260
32	Hartmann & Braun's, Frankfürt †	1896	Aperiodic	...	0·175	1600	
33	Crompton's Large pattern (Circular coil)	1896	12·1	Small	1·86	103·3	143·3
34	„ „ „ (Camb. Eng. Lab.) (Circular coil) .	1896	16·1	0·03	0·975	100	141
35	„ „ „ (W. R. Cooper, Esq.) („)	1897	7·2	0·018	0·51	62·6	68·5
36	„ Midget pattern (Camb. Eng. Lab.) („)	1897	10·0	Small	1·12	57·6	59·2
37	Nalder Ballistic (Camb. Eng. Lab.) (Rectangular coil)	1897	12·9	0·04	0·715	1053	1060
38	„ Damped No. 6283 (Rectangular coil)	1897	3·25	0·264	0·108	29·5	33·4
39	Andersen's Galvanometer (Circular coil)	1897	4·7	2·63	0·025	...	257
40	„ „ („)	1897	4·5	1·62	0·12	...	32·4
41	Ayrton-Mather Speaking Galvanometer (A) (Narrow coil) C.T.C.	1897	0·0965	...	0·0022	244	244
42	„ „ „ (B) („) „	1897	0·0868	...	0·0020	267	267
43	„ „ „ (C) („) „	1897	0·0925	...	0·0031	243	243
44	„ New pattern (Narrow coil) C.T.C.	1897	5·84	0·07	6·55	22·2	35·1
45	„ „ „ „	1897	7·6	...	17·7	1·9	5·75
46	Duddell's Oscillograph (Very narrow coil and electromagnet)‡	1897	0·000358	...	0·00012	1·3	1·3
47	„ „ („ „ „)‡	1898	0·000093	...	0·00020	2·08	2·08

* See Note § Table II.

† The constant of this instrument was kindly supplied by the makers.

‡ These instruments have their moving coils of phosphor bronze, whereas all the others are of copper or silver.

Explanation of Table V.

Column 1 (or T) gives the periodic time of vibration of coil when tested (in seconds).

„ 2 (or λ) „ logarithmic decrement of motion when tested.

„ 3 (or F′) „ deflexion in divisions per microvolt when scale-distance = 1000 scale-divisions.

„ 4 (or r) „ resistance of coil of instrument in ohms at about 15° C.

„ 5 (or r') „ resistance of instrument between terminals at about 15° C.

As will be seen from the table, considerable advance has been made in recent years. Of the d'Arsonval galvanometers mentioned in our 1890 paper, the most sensitive instrument as a voltmeter we had then seen was one of the "invariable sensibility" type, having a very large magnet. Its record is given in line 4 of the d'Arsonval section of that table (see line 34, Table I.). The instrument had a period of 2·6 seconds, coil-resistance of 21 ohms, and total resistance of 57·5 ohms, and gave a deflexion of 0·105 division per microvolt when the scale-distance was equal to 1000 scale-divisions. Referring to Table V. of the present paper, we notice that in 1892 a narrow-coil galvanometer was constructed, having a period of 2·2 seconds, coil-resistance 13·2 ohms, and total resistance 24·8 ohms, which gave 1·31 divisions per microvolt, an increase of over 12 times in sensitiveness, with a shorter period.

In 1893 Messrs. Queen and Co., of Philadelphia, produced a galvanometer of 178 ohms resistance, which gave 1·13 divisions per microvolt, whilst in 1896 Messrs. Crompton showed to the authors one of their high-grade instruments, having a total resistance of 143 ohms, which gave 1·83 divisions per microvolt, its period being 12·1 seconds (see line numbered 33, Table V.). Up to the end of 1896 the Crompton instrument was the most sensitive d'Arsonval voltmeter we had any record of, but its period was, for some purposes, inconveniently long.

Table VI. is an abstract of Table V. arranged chronologically, showing improvements since 1888.

TABLE VI.

Description.	Date.	Period in seconds.	Divisions per microvolt.	Resistance of coil in ohms.	Total resistance in ohms.
1. Invariable Sensibility Type (Large)	1888	2·6	0·105	21	57·5
2. Ayrton-Mather Narrow coil	1892	2·2	1·31	13·2	24·8
3. Queen and Co., Philadelphia	1893	Aperiodic.	1·13	.....	178
4. Crompton High Grade.	1896	12·1	1·83	103·3	143·3
5. Ayrton-Mather Narrow coil	1897	5·84	6·55	22·2	35·1
6. Ayrton-Mather Narrow coil	1897	7·6	17·7	1·9	5·75

During 1897 several fairly low-resistance d'Arsonval instruments of the narrow-coil type were made at the Central Technical College for use with the Lorenz apparatus tested by Prof. J. Viriamu Jones and one of the authors for the McGill University, Montreal. Records of two of them are given in lines numbered 44 and 45 of Table V. and lines 5 and 6 of Table VI. From these it will be seen that one gave 6·55 divisions per microvolt, and the other 17·7 divisions per microvolt, the latter number being nearly ten times as great as the previous best. Further particulars of these two very sensitive instruments when used as ammeters are given in lines 44 and 45 of Table II.

Uniformity in recording Tests.

We would here mention that in some of the galvanometer records we have received, the time of a single swing from one side of the scale to the other has been given as "the period," and this has caused considerable trouble on account of the sensibilities under standard conditions, calculated out on the assumption that the "complete period" was meant, coming out abnormally high. We therefore think it important that in all records of tests the "complete period," *i. e.* the time between two transits of the spot across the zero *in the same direction*, should be given. We may also remark that in all cases the logarithmic decrement (or the decrement) under the conditions in which the period was tested, should be

stated, so that the change of period caused by damping may be allowed for. Unless this be done, well-damped instruments, which are usually the most convenient, will be unfairly handicapped. In the case of d'Arsonval or other instruments, in which the periodic time cannot readily be altered by the user, it is important that the actual period of the vibration when tested should be given.

For a complete record the following particulars are required :—

(1) Resistance of instrument*.
(2) Periodic time when tested.
(3) Divisions deflexion per microampere (or microamperes per division).
(4) Scale-distance.
(5) Length of one scale-division.
(6) Logarithmic decrement under test conditions.
(7) Moment of inertia of the suspended system.
(8) Mass of suspended system.
(9) Diameter of mirror.
(10) Length of suspended magnets (perpendicular to axis of rotation).
(11) Type of instrument.
(12) Volume of coils.
(13) Date of test.

Nomenclature relating to Dead-beat Galvanometers.

Some ambiguity at present exists in the use of the words "dead-beat" as applied to galvanometers. Sometimes it is used as synonymous with "well-damped" or "aperiodic," whilst at other times the adjective is applied to instruments having quick-moving systems irrespective of whether the decrement of the motion is large or small. Since quickening the movement by increasing the control (*i. e.* decreasing the periodic time of vibration) of a given system lessens the decrement, the two uses of the words "dead-beat" are to some extent incompatible. Maxwell used the term for galvanometers in which the motion of the suspended system was "aperiodic,"

* If instrument be of the moving-coil type the resistances of coil and suspensions should be stated separately.

i. e. the system only passed once through the position of equilibrium before coming to rest. As this meaning of the word seems most rational, we think it desirable it should be retained, and that its use in connexion with instruments having rapidly moving systems whose motion is not aperiodic should be discontinued. The expressions "quick-moving" or "short-period" galvanometer might be used to denote this class of instrument.

Insulation of Coils and Terminals of Galvanometers, and the use of Price's "Guard-Wire."

When testing very high resistances, such as the insulation resistance of a short length of good cable, with a galvanometer, the difficulties which arise from leakage over the cable ends or from the galvanometer itself are well known; the importance of constructing galvanometers so as to ensure good insulation was referred to in our previous paper *.

Since that date Mr. W. A. Price † has suggested the use of a "guard-wire" which practically eliminates error due to surface leakage over cable ends. The method there described can be adapted to galvanometers, thereby making the question of *perfect* insulation of coils and terminals of far less consequence than formerly.

Mr. Price's method consists in winding a bare wire W about midway between the extremities of the pared ends of the cable as shown in fig. 1, and connecting the other end of the wire to H, the battery side of the galvanometer G.

As the current through the galvanometer is very small, the P.D. between the wire W and the conductor C is also very small, and the surface leakage between them is quite negligible if the surface of the dielectric between them is fairly clean and dry. The full P.D. exists between W and the sheathing or wet braiding S, but any leakage-current from one to the other does not pass through the galvanometer. In the same

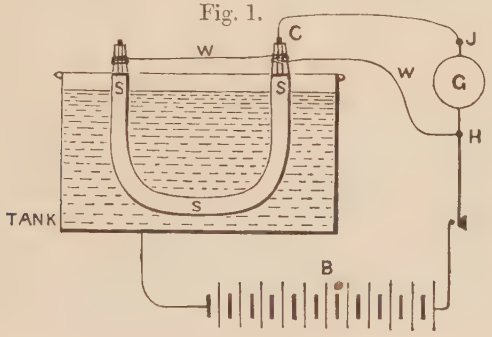
* See p. 71, Phil. Mag. July 1890.

† 'Electrical Review,' vol. xxxvii. p. 702 (1895). See also Appleyard on "Dielectrics," Proc. Phys. Soc. vol. xiv. pp. 257, 264, where possible error is evaluated.

manner error due to leakage from the terminal J of the galvanometer may be eliminated.

It will be evident that there is no need to have the battery side of the galvanometer *perfectly* insulated : all that is necessary is that the insulation resistance of H from earth is large compared with the internal resistance of the battery.

Fig. 1.



Probably the simplest way of applying the "guard-wire" principle to a galvanometer is to put one terminal of the winding to frame and case (preferably of metal), and carry the other terminal from the frame on a support moderately well insulated therefrom. The whole galvanometer may be sufficiently well insulated from earth by putting short ebonite sleeves, or caps, on the tips of the levelling screws. This construction was carried out in the Ayrton-Mather galvanometer as made by Mr. R. W. Paul in 1892, an instrument of this class being shown before this Society on June 10th in that year *. By using the frame terminal as the battery side of the galvanometer, the frame acts as an efficient guard-wire. Another advantage of the construction here described is that electrostatic deflexion of the moving system is at the same time prevented, a matter of considerable importance when resistance tests with fairly large potential-differences are being made.

A similar construction is, of course, applicable to shunt-boxes.

In concluding this section of our paper, we desire to thank

* 'Electrician,' vol. xxix. June 17th, 1892, p. 174.

the various manufacturers and inventors who have kindly supplied us with information and data relating to their instruments, and also to acknowledge our great indebtedness to C. G. Lamb, Esq., M.A., of the Cambridge Engineering Laboratory, and W. R. Cooper, Esq., M.A., for supplying us with several important records by which we have been able to add materially to the completeness of our lists.

Section B.

Limiting Sensitiveness of Thomson Galvanometers.

At the Oxford meeting of the British Association (1894), Prof. Schuster read a paper "On the Construction of Delicate Galvanometers," in which inquiry is made as to the smallest current that can be detected by a non-astatic instrument of 1 ohm resistance, and controlled by a field of strength 0·17 C.G.S. unit. The calculated minimum current depends on the resolving power of the mirror employed, the smallest angle observable being taken as $\frac{\lambda}{2d}$ where λ is the wave-length of light and d the diameter of the mirror. The result arrived at is that $1{\cdot}5 \times 10^{-8}$ amperes is the smallest current that can be detected under the assumed conditions.

Taking λ as $5{\cdot}9 \times 10^{-5}$ centimetres we find the minimum angle to be $2{\cdot}95 \times 10^{-5}$ radians or about 6·1 seconds of arc; this corresponds with 0·0593 division on a scale at 1000 divisions distance. Expressing Prof. Schuster's result in this way we get 3·9 divisions per microampere as the limiting sensibility of a non-astatic galvanometer of 1 ohm resistance when controlled by a field of 0·17 C.G.S. unit, the cavity within the coil being a sphere of 1 centimetre diameter. It is interesting to notice that to obtain a deflexion of 200 divisions per microampere (a number one might fairly expect to get from a good Thomson instrument having a period of 10 seconds and needles about a centimetre long) would necessitate the controlling field of a non-astatic instrument being reduced to less than one-fiftieth of the assumed value, or to 0·003 C.G.S. unit.

In his calculations Prof. Schuster had no need to take into account the period of vibration of the suspended system, but

in the following treatment of the subject period is considered an important factor, and the specific magnetism of the needle is also taken into account. Further, the suspended system is supposed astatic, but this is not essential, for a non-astatic system with an adjustable control would lead to the same results.

The calculations in this section of the paper were made to ascertain whether some unusually high records of galvanometer tests which had been sent to us could possibly be correct. The answer was in the affirmative. A similar investigation had previously been made relating to some instruments of the d'Arsonval type, and this gave an answer in the negative. The calculated maximum sensibility came out about $\frac{1}{20}$ of the published value. Subsequent tests made on an actual instrument gave a result about $\frac{1}{3}$ of the published value, or about two-thirds of the calculated maximum.

In treating the subject mathematically the following assumptions are made :—

(1) Clearance between coil and end of magnets 1 millimetre.
(2) Mirror and stem of negligible inertia compared with that of magnets.
(3) Perfect astaticism of needle. (Thickness negligible.)
(4) Coil unlimited in size and of best shape as voltmeter.
(5) Thickness of insulation on wire negligible.
(6) Control of suspension negligible.
(7) Deflecting field equal to field at centre of coil.

Note.—All the assumptions except (7), and perhaps (1), will give too high a value for the calculated sensitiveness.

Let $2l$ = length of magnets.
A = area of cross-section of magnets.
B = induction density in the steel.
Δ = density of steel.
m = strength of pole.
σ = specific magnetism of needles.
τ = time of vibration (complete).

Then,

$$\text{Magnetic moment} = 2ml,$$
$$= M \text{ (say)},$$
$$\therefore \quad M = \frac{2AB l}{4\pi}; \qquad \ldots \quad (8)$$

$$\text{Mass} = 2Al\Delta,$$

$$\text{Moment of inertia } I = \tfrac{2}{3}Al^3\Delta; \quad . \quad . \quad . \quad . \quad . \quad (9)$$

$$\text{Specific magnetism} = \frac{B}{4\pi\Delta}; \quad . \quad . \quad . \quad . \quad . \quad (10)$$

$$\therefore \quad \sigma = \frac{B}{98} \text{ approximately.}$$

For an astatic pair we have

$$\tau = 2\pi\sqrt{\frac{2I}{MH_1 - MH_2}}$$

where H_1 and H_2 are the controlling fields acting on the two magnets respectively;

$$\therefore \quad H_1 - H_2 = \frac{8\pi^2 I}{\tau^2 M}$$

$$= \frac{8}{3}\frac{\pi^2 l^2}{\tau^2 \sigma} \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

from (8), (9), and (10).

This gives the difference of the two fields for any periodic time of vibration, length of needle, and specific magnetism.

If θ be the deflexion produced by a current γ, then

$$\text{Controlling couple} = (H_1 - H_2)\, M \sin\theta,$$

and $$\text{Deflecting couple} = 2M\gamma G \cos\theta,$$

where G is the principal constant of the coil.

Hence

$$\gamma = \frac{(H_1 - H_2)}{2G}\tan\theta,$$

$$= \frac{4\pi^2 l^2}{3\tau^2 \sigma G}\tan\theta \quad . \quad . \quad . \quad . \quad (12)$$

from (11).

Now

$$G = N\frac{1}{a^2\beta^2}\left(\frac{1}{a} - \frac{1}{x}\right) \quad . \quad . \quad . \quad (13)$$

(see Maxwell, vol. ii. p. 325, 1st ed.);

where

$$N = 2\pi\int_0^{\pi} (\sin\theta)^{\frac{1}{2}}\, d\theta$$

$$= 9{\cdot}034.$$

α the ratio of the radius of the wire to the maximum radius of the layer;

β the ratio of the distance between the centre of two adjacent wires to the radius of the wire;

a the maximum radius of cavity in coil;

x the outer radius of coil.

The resistance of the coil is given by

$$R = N \frac{\rho}{\pi \alpha^4 \beta^2} \left(\frac{1}{a} - \frac{1}{x} \right), \quad . \quad . \quad (14)$$

ρ being the specific resistance of metal used in the winding;

$$\therefore \quad \frac{G}{\sqrt{R}} = \frac{1}{\beta} \sqrt{N \frac{\pi}{\rho} \left(\frac{1}{a} - \frac{1}{x} \right)} \quad . \quad . \quad (15)$$

A little consideration will show that $\frac{G}{\sqrt{R}}$ gives the value of G for a galvanometer whose resistance is one ohm.

This is a maximum when $\beta = 2$ and $x = \infty$.

Since

$$a = l + 0{\cdot}1 \quad . \quad . \quad \{\text{assumption (1)}\}$$

we get

$$\gamma = \frac{4\pi^2 l^2 \beta}{3\tau^2 \sigma \sqrt{N \frac{\pi}{\rho} \frac{1}{l + 0{\cdot}1}}} \tan \theta \quad \text{from (12) and (15)}$$

$$= \frac{4\pi^2 l^2 \beta}{3\tau^2 \sigma} \sqrt{\frac{\rho (l + 0{\cdot}1)}{N\pi}} \tan \theta \quad . \quad . \quad . \quad . \quad (16)$$

Now for 1 division deflexion at a scale distance of 1000 divisions, $\theta = \frac{1}{2000}$, so taking

$$\tau = 10 \text{ seconds},$$
$$\rho = 1{\cdot}64 \times 10^{-6}, \text{ and}$$
$$\sigma = 50,$$

corresponding with B = 5000 approximately, a high value for short permanent magnets, we get

$$\text{Current per 1 division} = \frac{6{\cdot}32}{10^{10}} l^2 \sqrt{l + 0{\cdot}1} \text{ C.G.S. units},$$

$$= \frac{6{\cdot}32}{10^{9}} l^2 \sqrt{l + 0{\cdot}1} \text{ amperes} \quad . \quad (17)$$

From this formula the following table has been calculated.

TABLE VII.

Half length of needles (l) in centimetres.	Amperes per division.	Divisions per Microampere.
0·05	$6·12\times10^{-12}$	163,000
0·1	$2·86\times10^{-11}$	35,000
0·15	$7·16\times10^{-11}$	14,000
0·2	$1·38\times10^{-10}$	7,250
0·3	$3·60\times10^{-10}$	2,775
0·4	$7·14\times10^{-10}$	1,400
0·5	$1·22\times10^{-9}$	820

With these results before us it is interesting to examine how nearly the limits here given have been approached in actual instruments.

One galvanometer for which the dimensions of the magnets are known is that mentioned in line 9 of Table II. In this instrument the magnets are 0·82 centimetre long, therefore $l=0·41$. From Table VII. an instrument with such a needle and wound to have a resistance of 1 ohm should give about 1350 divisions per microampere under the assumed conditions. On referring to Table II., line 9, we find the value of $D/r^{\frac{3}{5}}$ (the expression most suited for a high-resistance instrument with small coils) actually obtained to be 380, or little more than a quarter $\left(\frac{1}{3·56}\right)$ the calculated possible value.

Another comparison can be made in the case of Mr. McKittrick's oscillograph, which has a needle of almost microscopic dimensions. In this instrument $l=0·053$, and we see from Table VII., that we might possibly get 150,000 divisions per microampere when the resistance is one ohm, period 10 seconds, and scale-distance equals 1000 scale-divisions. Line 4 of Table II. gives the values 60,000 and 105,000 for $D/r^{\frac{1}{5}}$ and $D/r^{\frac{2}{5}}$ respectively. These numbers approximate more closely to the theoretical values than those for the Mudford galvanometer previously referred to, the probable reason being that the needle of Mr. McKittrick's

instrument consists of a piece of soft iron placed in a very strong magnetic field. The specific magnetism would therefore be much greater than that of short permanent magnets in a weak field such as used in controlling the needle of the Mudford galvanometer. In all probability the value of σ for Mr. McKittrick's instrument would be two or three times as great as the value used in calculating Table VII.

Section C.

Long versus *Short Period Galvanometers for Zero Methods.*

For rapidity and accuracy of working a short-period instrument is certainly better than a long-period one, providing its sensitiveness is as great. It is interesting, however, to inquire whether, with the same galvanometer, it is more expeditious to use a strong or a weak control, provided the sensitiveness when strongly controlled is sufficient for the purpose. For although the spot will move to its maximum elongation, corresponding with a given want of balance, more quickly when strongly controlled, that elongation will be less than when a weak control is used. Hence it is possible that a given small displacement of the spot from the zero position may occur in a shorter time when the control is weak than when it is strong, because such displacement is then a smaller fraction of the whole.

First applying general reasoning to the problem, we may notice that the deflecting couple produced by a given want of balance is initially independent of the control, and as the inertia of the moving system is constant, the acceleration at the beginning of the motion is the same whatever the control. If f be this acceleration the space traversed by the spot in a short time t is $\frac{1}{2}ft^2$, and this is therefore independent of the control, consequently for small displacements strong or weak control makes no difference in the time it takes to see that a want of balance exists.

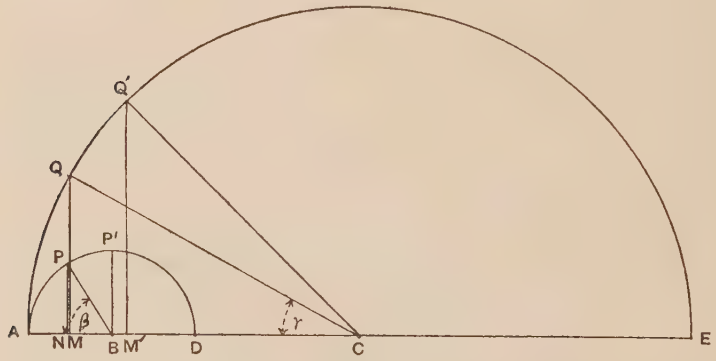
It is to be noted, however, that the weakly controlled system will take longer to return to zero, so that a longer wait is necessary before another test can be made. On the other hand, should the sensitiveness of the strongly controlled instrument be only just sufficient to get the desired accuracy,

then half a period must elapse before the want of balance can be detected, whereas the weakly controlled instrument would have moved over a greater distance in that time. Taking the controls in the ratio of 4 to 1, and therefore periods as 1 to 2, the movement of the weakly controlled instrument will be double that of the other in the case here considered. In this instance therefore the long periodic time is advantageous because it will show a very small want of balance more quickly, but the advantage is somewhat discounted by the longer time occupied in the return to zero.

Undamped Motion.

To consider the matter in further detail let us first take the case of undamped simple harmonic motion. Assuming the ratio of the controls to be as m^2 to 1, the periods will be in the ratio of 1 to m.

Fig. 2.



Draw two circles touching at A, centres at B and C, and of radii in the ratio of 1 to m^2. Draw lines BP and CQ making angles β and γ with AC and such that $\beta = m\gamma$. The distances from A of N and M, the projections on AC of P and Q respectively, will then represent the displacements from zero after a given time of the short and long period galvanometers respectively. From the figure (to scale) it will be seen that these are nearly equal when the time is, say, one-sixth of a period of the quick-moving system. And even when the time is one-fourth of the period of the strongly controlled

instrument, the difference in the displacements is not very marked, for in that time the short-period needle will have moved over the space AB, whilst the long-period needle will have traversed the distance AM′.

Again,

$$\mathrm{AN}=\mathrm{AB}(1-\cos\beta),$$
$$=b(1-\cos\beta) \quad \text{(say)},$$

and

$$\mathrm{AM}=\mathrm{AC}(1-\cos\gamma),$$
$$=c(1-\cos\gamma) \quad \text{(say)};$$

but $c=m^2b$,

and $\beta=m\gamma$,

$$\therefore \quad \mathrm{AN}=b(1-\cos m\gamma),$$

and

$$\mathrm{AM}=m^2b(1-\cos\gamma).$$

$$\therefore \quad \frac{\mathrm{AM}}{\mathrm{AN}}=\frac{m^2(1-\cos\gamma)}{1-\cos m\gamma}. \qquad \ldots\ldots \quad (1)$$

Expanding the cosines we get

$$\frac{\mathrm{AM}}{\mathrm{AN}}=\frac{m^2\left\{\frac{\gamma^2}{\underline{|2}}-\frac{\gamma^4}{\underline{|4}}+\frac{\gamma^6}{\underline{|6}}-\&c.\right\}}{\left\{\frac{m^2\gamma^2}{\underline{|2}}-\frac{m^4\gamma^4}{\underline{|4}}+\frac{m^6\gamma^6}{\underline{|6}}-\&c.\right\}}.$$

Dividing top and bottom by $\frac{m^2\gamma^2}{\underline{|2}}$, we have

$$\frac{\mathrm{AM}}{\mathrm{AN}}=\frac{1-\frac{\gamma^2}{12}+\frac{\gamma^4}{360}-\&c.}{1-\frac{m^2\gamma^2}{12}+\frac{m^4\gamma^4}{360}-\&c.}. \qquad \ldots \quad (2)$$

This, to the 1st degree of approximation, is unity, and only differs slightly from unity when $m\gamma$ is considerable, thus confirming the general reasoning above. Taking m equal 2, as in the figure, and γ equal to $\frac{1}{2}$, *i. e.* 28°·6, we have

$$\frac{\mathrm{AM}}{\mathrm{AN}}=\frac{1-\frac{1}{48}+\frac{1}{5760}-\&c.}{1-\frac{1}{12}+\frac{1}{360}-\&c.}$$
$$=1{\cdot}06.$$

Consequently the displacements differ by about 6 per cent. in a time equal to about one-sixth $\left(\frac{1}{2\pi}\right)$ of the period of the quick-moving instrument.

To put the matter in another way, we may determine the relative times after which the spots will have moved appreciably from the zero. A little consideration will show that these times will depend on the relation of the least visible movement to the maximum displacement produced by the want of balance it is necessary to detect. For example, if the sensitiveness of the short-period instrument is such as to give a fairly large movement of the spot for the given want of balance, then a perceptible movement will occur in a time which is a small fraction of its period, and this motion will take place in practically the same time as an equal motion of the spot of the long-period instrument. On the other hand, if the sensitiveness of the short-period instrument is only just sufficient to show the given want of balance when the spot reaches its maximum elongation, then half a period must elapse before the want of balance would be detected and the more sensitive long-period instrument would have moved an equal distance in a shorter time. In this case therefore, the more sensitive instrument would have a distinct advantage in quickness in showing the want of balance, but more time would be required for its return to zero before another test could be made.

To show the relative times under different conditions of required accuracy, and of different relative periods of similar galvanometers, we have calculated the following table.

The first column gives a measure of the sensitiveness of the short-period instrument in terms of the minimum possible value consistent with the desired accuracy, *e.g.*, the 1 in the first column indicates that the instrument is *only just sensitive enough* to indicate the maximum permissible error in balance ; the 2 in the first column denotes that the instrument is twice as sensitive as absolutely necessary ; and the 3 that it has three times the necessary sensibility &c. The numbers in line 1 below the various values of m^2 show the relative times required for the spot to move perceptibly, the time occupied by the slow-moving instrument in moving over the minimum

TABLE VIII.

Ratio of maximum displacement of short-period needle to least movement visible.	Ratio of $\frac{\text{Time taken by short-period galv. to move visibly}}{\text{" " longer " " "}}$ when the sensitiveness of the longer-period instrument is m^2 times as great as that of the short-period instrument.				
	$m^2=2.$	$m^2=4.$	$m^2=16.$	$m^2=64.$	$m^2=10{,}000.$
1	1·41	1·50	1·55	1·60	1·63
2	1·06	1·09	1·10	1·10	1·11
3	1·034	1·05	1·053	1·054	1·055
4	1·025	1·04	1·045	1·047	1·048
8	1·011	...	...	...	1·021
10	1·01	...	...	...	1·019
20	1·005	...	...	...	1·007

perceptible distance being taken as unity. In column 2 line 1 we find the number 1·41, and this indicates that if the short-period instrument is only just sensitive enough to show the maximum permissible want of balance, then, doubling the sensitiveness by making the period $\sqrt{2}$ times as great, would enable the want of balance to be detected in a shorter time, the times being in the ratio of 1 to 1·41. Similarly for column 3 line 1, increasing the period to twice that of the short-period instrument and therefore increasing the sensitiveness four times would reduce the relative times to the ratio of 1 to 1·5. When m equals 100 (or the sensitiveness is increased ten thousand times, if such an increase were possible) the relative times become 1 to 1·63, showing a comparatively small extra diminution in time required to detect a want of balance, although the sensibility is very largely increased.

The numbers in line 2 show that when the short-period instrument is twice as sensitive as is absolutely necessary the gain in time required to detect the want of balance becomes less, the ratio of the times being 1 : 1·06 when m^2 equals 2, and only reducing to a value 1 : 1·11 when m^2 equals 10,000. Subsequent lines show that when the short-period

galvanometer is several times more sensitive than is absolutely necessary to obtain the desired accuracy, the gain in quickness in detecting a small error by weakening the control becomes unimportant, and so the small gain will be more than neutralized by the longer time taken in the return to zero.

Aperiodic Motion.

If the motion of the suspended system be aperiodic, as in well-damped d'Arsonvals, the problem is rather more complicated.

When damping forces act on a vibrating system the equation of motion is

$$\frac{d^2\theta}{dt^2}+\frac{2B}{I}\frac{d\theta}{dt}+\frac{C}{I}\theta=0,$$

$$\therefore\quad \theta=A_1\,\epsilon^{-(p+q)t}+A_2\epsilon^{-(p-q)t},$$

where

$$p=\frac{B}{I},\text{ and }q^2=\frac{B^2}{I^2}-\frac{C}{I},$$

$$\text{or}\quad q^2=p^2-n^2,\text{ where }n^2=\frac{C}{I}.$$

Let $\theta=\alpha$ when $t=0$.

Also $\dfrac{d\theta}{dt}=0$ „ $t=0$,

then

$$\theta=\frac{\alpha}{2q}\epsilon^{-pt}\{(p+q)\epsilon^{qt}-(p-q)\epsilon^{-qt}\}.$$

If we now lessen the control to $\dfrac{1}{m^2}$ its initial value, so that the period, if undamped, would be lengthened m times, the sensitiveness will increased m^2 times, and consequently the value of α is increased m^2 times. This change of C alters q, but p is not affected thereby. Let the new value of q be q'.

Under the altered conditions we have

$$\theta'=\frac{\alpha'}{2q'}\epsilon^{-pt}\{(p+q')\epsilon^{q't}-(p-q')\epsilon^{-q't}\}.$$

We may also write θ in the form

$$\theta=\frac{\alpha}{2q}\{(p+q)\,\epsilon^{-(p-q)t}-(p-q)\,\epsilon^{-(p+q)t}\},$$

and expanding

$$\theta=\frac{\alpha}{2q}\{(p+q)[1-(p-q)t+\frac{(p-q)^2t^2}{\lfloor 2}-\&c.]$$
$$-(p-q)[1-(p+q)t+(p+q)^2t^2-\&c.]\}$$
$$=\frac{\alpha}{2q}\{2q+(p^2-q^2)\frac{t^2}{\lfloor 2}[p-q-(p+q)]$$
$$-(p^2-q^2)\frac{t^3}{\lfloor 3}[(p-q)^2-(p+q)^2]+\&c.\}$$
$$=\frac{\alpha}{2q}\{2q+(p^2-q^2)\frac{t^2}{\lfloor 2}(-2q)-(p^2-q^2)\frac{t^3}{\lfloor 3}(-4pq)+\&c.\}$$
$$=\alpha\{1-(p^2-q^2)\left[\frac{t^2}{\lfloor 2}-2p\frac{t^3}{\lfloor 3}+\text{terms involving } t^4 \text{ \&c. } p \text{ and } q\right]$$
$$=\alpha\{1-n^2\left[\frac{t^2}{\lfloor 2}-2p\frac{t^3}{\lfloor 3}+\text{terms \&c.}\right]$$
$$\therefore \theta-\alpha=-\alpha n^2\left[\frac{t^2}{\lfloor 2}-2p\frac{t^3}{\lfloor 3}+\text{terms \&c.}\right].$$

Similarly

$$\theta'-\alpha'=-\alpha' n'^2\left[\frac{t^2}{\lfloor 2}-2p\frac{t^3}{\lfloor 3}+\text{terms involving } t^4 \text{ \&c. } p \text{ and } q'\right].$$

Hence the ratio of the two displacements is

$$\frac{\theta'-\alpha'}{\theta-\alpha}=\frac{\alpha' n'^2}{\alpha n^2}\frac{\left[\frac{t^2}{\lfloor 2}-2p\frac{t^3}{\lfloor 3}+\text{terms in } t^4 \text{ \&c. } p \text{ and } q'\right]}{\left[\frac{t^2}{\lfloor 2}-2p\frac{t^3}{\lfloor 3}+\text{terms in } t^4 \text{ \&c. } p \text{ and } q\right]},$$

but $$\alpha'=m^2\alpha,$$

and $$n'^2=\frac{1}{m^2}n^2,$$

$$\therefore \quad \alpha' n'^2=\alpha n^2.$$

$$\therefore \quad \frac{\theta'-\alpha'}{\theta-\alpha}=1\times\frac{[\qquad\qquad]}{[\qquad\qquad]}.$$

Thus, when $\frac{t^4}{\lfloor 4}$ and higher powers are small, the ratio is unity, consequently there is no difference in the rapidity of detecting want of balance when the sensitiveness of the short-period instrument is sufficient for the purposes of the experiment.

To find the ratio in the other cases we may observe that the general term in t^r is, when r is odd,

$$-\frac{t^r}{\lfloor r}\{(r-1)p^{r-2}+\frac{(r-1)(r-2)(r-3)}{\lfloor 3}p^{r-4}q^2+\ldots+(r-1)pq^{r-3}\};$$

and when r is even

$$+\frac{t^r}{\lfloor r}\{(r-1)p^{r-2}+\frac{(r-1)(r-2)(r-3)}{\lfloor 3}p^{r-4}q^2+\ldots+q^{r-2}\}.$$

Decreasing the control decreases n^2 and therefore increases q,

$$\because \quad q^2=p^2-n^2.$$

Hence q' is $>q$,

consequently the term in t^4 in the numerator of $\frac{\theta'-\alpha'}{\theta-\alpha}$ will be greater than the corresponding term in the denominator, and this indicates that the displacement of the slow-moving instrument gains on that of the quick-moving one as the time increases.

The above investigation teaches us that when we have a galvanometer whose control can be readily altered and whose sensitiveness can be easily made ample for the purpose of the test, then *for rapid working adjust the control so that the sensitiveness is, say, two or three times as great as is absolutely necessary for the desired accuracy.*

Discussion.

Prof. Threlfall thought the authors' method of comparing galvanometers very misleading. The results obtained in their comparison of the oscillograph (3,310,000) and the suspended-coil galvanometer (27) might be regarded as the *reductio ad absurdum* of the proposed system. The absurdity arose from the dissimilarity of the two instruments. Moreover, the proposed system ignored the fact that sensitiveness may be obtained by optical as well as by electromagnetic means. Optical sensitiveness, owing to its greater stability, was to be preferred to electromagnetic sensitiveness. The fundamental problem in the construction of galvanometers is an optical one; it is necessary to decide the mass and dimensions of the suspended parts so as to ensure (1) optical accuracy, and (2) electromagnetic sensitiveness. Thus, to some extent, the weight of the mirror determines the thickness of the suspension. As an instance of what might be done by optical methods, Prof. Threlfall referred to work done by himself and Mr. Brearley (Phil. Mag. 1896), in which it was possible to measure to $1{\cdot}48 \times 10^{-13}$ amperes, and, with special refinements, to 3×10^{-14} amperes. He had found that the best diameter for glass mirrors was 1·1 centim., with a weight just under 0·5 grm. These were used with a scale at 276 centim., read by a microscope to 0·04 millim. The course of the light was:—Lamp, large lens, small scale, mirror, eyepiece. The period was 25 secs., and the resistance 50,000 ohms. Even better results could be obtained by using mirrors of quartz or of blood-stone. Quartz is incomparably to be preferred to glass. Such figures indicated what could be done by optical sensitiveness—the sensitiveness that the authors ignored. It was pointed out by Prof. Threlfall that the controlling field for galvanometers of the "Thomson" type should be uniform. This was best secured by using two magnets, one above, and one below the needles.

Prof. Perry said the authors had not asserted that a galvanometer with higher figure of merit, according to their classification, was *superior* to another of lower figure. It must be agreed that the figure they obtain is a very

valuable datum for the comparison of instruments designed for similar purposes, for instance in classifying those used by Prof. Threlfall. Mr. Duddell was to be congratulated on the extreme sensitiveness and small period of his oscillograph.

Prof. AYRTON, referring to Prof. Threlfall's *reductio ad absurdum*, admitted that the criticism would carry some conviction if the two instruments were of different kinds; if, for instance, one possessed a suspended needle and the other a suspended coil. But the argument failed, because both instruments were of the suspended-coil type. In one of them Mr. Duddell had developed the advantages to be gained by reducing the air-gap. To form an opinion of electromagnetic improvements in galvanometers, it was necessary to reduce the results of all instruments to some system of classification. There was no objection, after that, to adding a good mirror, and reading by a good microscope.

ERRATUM.

On page 1 of this volume the title of the paper by Prof. Stroud should read:

"*A Telemetrical Spherometer and Focometer.*"

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